1. Prove that if $|G| = 105$, then $G$ has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup. Furthermore, if a Sylow 3-subgroup of $G$ is normal, then $G$ is abelian.

**Proof:** Suppose $|G| = 105 = 3 \cdot 5 \cdot 7$. For $p = 3, 5, 7$ let $n_p$ denote the number of Sylow $p$-subgroups of $G$. Recall by Sylow’s Theorem that if $|G| = p^a \cdot m$ where $p \nmid m$, then $n_p|m$ and $n_p \equiv 1 \mod p$. Therefore, in this case the possibilities for $n_p$ are

\[
\begin{align*}
n_7 &= 1 \text{ or } 15 \\
n_5 &= 1 \text{ or } 21 \\
n_3 &= 1 \text{ or } 7
\end{align*}
\]

Also, by Sylow’s Theorem there exists a unique Sylow $p$-subgroup if and only if a Sylow $p$-subgroup is normal in $G$ if and only if a Sylow $p$-subgroup is characteristic in $G$. Let $P \in \text{Syl}_7(G)$, $Q \in \text{Syl}_5(G)$, and $R \in \text{Syl}_3(G)$.

We claim that at least one of $n_7$ or $n_5$ must equal 1. Otherwise, we have that $n_7 = 15$ while $n_5 = 21$. Notice that By Lagrange’s Theorem any Sylow 7-subgroup has order 7 and also any two different Sylow 7-subgroups have trivial intersection. It follows that each Sylow 7-subgroup yields 6 distinct elements of order 7. Since there are 15 different Sylow 7-subgroups this produces $15 \cdot 6 = 80$ elements of order 7 in $G$. Similarly, we gather that there are $21 \cdot 4 = 84$ different elements of order 5 in $G$. But this says that $G$ has at least 164 different elements; a contradiction. Therefore, either one of $n_7$ or $n_5$ must equal and there either a $P \trianglelefteq G$ or $Q \trianglelefteq G$.

Consider $PQ \subseteq G$. Since one of $P$ or $Q$ is normal in $G$ it follows that $PQ \leq G$. Once again we apply lagrange’s Theorem to conclude that $P \cap Q = \{e_G\}$ and so since

$$|PQ| = \frac{|P||Q|}{|P \cap Q|}$$

we obtain that $|PQ| = 35$. Furthermore,

$$|G : PQ| = \frac{|G|}{|PQ|} = 3$$

which is the least prime dividing the order of $G$. By theorem this means that $PQ \trianglelefteq G$. Applying the counting technique used above to the group $PQ$ yields that $P \trianglelefteq PQ$ and $Q \trianglelefteq PQ$. But a Sylow subgroup of a group is normal if and only if it is characteristic in said group. Therefore, $P \trianglelefteq G$ and $Q \trianglelefteq G$. It is straightforward to check that a characteristic subgroup of a normal subgroup is normal in the ambient group. Therefore, both $P, Q \trianglelefteq G$.

Similar arguments to the ones we already used prove that $PQ \cap R = \{e_G\}$ and $(PQ)R \leq G$, and so by order considerations $G = (PQ)R$. Since $PQ \trianglelefteq G$ it follows that at this point we can recognize $G$ as a semi-direct product and conclude that

$$G \cong (PQ) \rtimes R$$

For some homomorphism $\phi : R \to \text{Aut}(PQ)$. This same argument can be used to recognize that $PQ \cong P \times Q \cong \mathbb{Z}_7 \times \mathbb{Z}_5$ which is abelian.

Now, if we also assume that $n_3 = 1$ (i.e. $R \cong G$), then we know that $\phi$ is the identity map and so

$$G \cong (PQ) \times R \cong \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_3$$

which is an abelian group. ■

2. Let $G$ be a finite group acting on the set $A$. Suppose that $H \normaleq G$ so that for any $a_1, a_2 \in A$ there is a unique $h \in H$ so that $h \cdot a_1 = a_2$. For each $a \in A$, recall that $G_a$ is the stabilizer of $a$ in $G$. Prove
(a) $G = G_a H$ and $G_a \cap H = \{ e_G \}$.

(b) if $H$ is contained in $Z(G)$, then $G_a$ is a normal subgroup and $G$ is isomorphic to $G_a \rtimes H$.

**Proof:** (a). let $a \in A$. First observe that by hypothesis for any $a_1, a_2 \in A$ there is a unique element of $H$, say $h \in H$, such that $h \cdot a_1 = a_2$. Applying this to $a_1 = a = a_2$ and knowing that $e_G \in H$ and $e_G \cdot a = a$ it follows that $e_G$ is the only element of $H$ which stabilizes $a$. Therefore, $G_a \cap H = \{ e_G \}$. Since $H$ is normal it follows that $G_a$ normalizes $H$ and so $HG_a = G_a H \leq G$. To show that $G = HG_a$ it suffices to show that for any $g \in G$ there is an $x \in G_a$ and an $h \in H$ such that $g = hx$. Let $b = g \cdot a$. By hypothesis there is a unique $h \in H$ such that $h \cdot b = a$. Then,

$$(hg) \cdot a = h \cdot (g \cdot a) = h \cdot b = a$$

which means that $hg \in G_a$. Since $h \in H$ and $H$ is a subgroup $h^{-1} \in H$. Therefore, since $g = h^{-1}(hg)$ with $h^{-1} \in H$ and $hg \in G_a$ it follows that $G = HG_a$.

(b). Recognizing that for any $a \in A$ that $H \subseteq G$, $G_a \cap H = \{e_G\}$, and $G = HG_a$ we conclude that $G \cong H \rtimes \phi$ where $G_a \to \text{Aut}(H)$. Next, we suppose that $H \leq Z(G)$. Then notice that for any $g \in G$, we have $g = hx$ for some $h \in H$ and $x \in G_a$. Therefore,

$$gG_ag^{-1} = (hx)G_a(xh)^{-1} \quad (4)$$
$$= (hx)G_a(x^{-1}h^{-1}) \quad (5)$$
$$= h(xG_ax^{-1})h^{-1} \quad (6)$$
$$= hG_a h^{-1} \quad (7)$$
$$= G_a \quad (8)$$

where (7) holds since $x \in G_a$, and (8) holds since $h \in Z(G)$. Consequently, $G_a \leq G$. Since we already know that $G \cong H \rtimes \phi G_a$, now the normality of $G_a$ implies that $G \cong H \rtimes G_a$.■

3. Find all finite groups which have exactly two conjugacy classes.

**Proof:** Suppose $G$ is a finite group with exactly two conjugacy classes and let $n = |G|$. Let $K_1, K_2$ be the two conjugacy classes. Without loss of generality we suppose that $e_G \in K_1$. Then since the conjugacy class of a central element is the singleton set containing that element we know that $K_1 = \{e_G\}$. It follows that $K_2 = G \setminus \{e_G\}$ and so $|K_2| = n - 1$. Recall that the proof of the Class Equation allows us to conclude that the size of each conjugacy class is a divisor of the order of the group. Therefore, $n - 1 | n$. But this means that $n = 2$. Consequently, $G \cong \mathbb{Z}_2$.■