Quasiconformal maps on non-rigid Carnot groups

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Abstract

We study quasiconformal maps on non-rigid Carnot groups equipped with Carnot metric. We show that for most non-rigid Carnot groups $\mathcal{N}$, all quasiconformal maps on $\mathcal{N}$ must be biLipschitz.

Keywords. rigid Carnot groups, non-rigid Carnot groups, quasiconformal maps, quasisymmetrically rigid.


1 Introduction

We study quasiconformal maps on non-rigid Carnot groups equipped with Carnot metric. We show that even most non-rigid Carnot groups are rigid with respect to quasiconformal map. To be more precise, for most non-rigid Carnot groups $\mathcal{N}$, all quasiconformal maps on $\mathcal{N}$ must be biLipschitz.

A Carnot group $\mathcal{N}$ is called $C^2$-rigid if the space of $C^2$ contact maps between domains in $\mathcal{N}$ is finite dimensional, and is called non-rigid otherwise. In particular, the space of $C^2$ quasiconformal maps on rigid Carnot groups equipped with Carnot metric is finite dimensional. On the other hand, if $\mathcal{N} = V_1 \oplus \cdots \oplus V_r$ is the Lie algebra of $\mathcal{N}$ and there exists some nonzero $X \in V_1$ such that the linear map $ad X : \mathcal{N} \to \mathcal{N}$, $ad X(Y) = [X,Y]$ has rank at most one, then the space of biLipschitz maps from $\mathcal{N}$ to $\mathcal{N}$ is infinite dimensional (see Proposition 6.3). Ottazzi and Warhurst ([OW], Theorem 1) showed that a Carnot group is non-rigid if and only if the complexification $\mathcal{N} \otimes \mathbb{C}$ of the Lie algebra $\mathcal{N}$ of $\mathcal{N}$ has an element of rank at most one.

Let $K \geq 1$ and $C > 0$. A bijection $F : X \to Y$ between two metric spaces is called a $(K,C)$-quasi-similarity if

$$\frac{C}{K} d(x,y) \leq d(F(x),F(y)) \leq C K d(x,y)$$

for all $x,y \in X$.

*Partially supported by NSF grant DMS-1265735.
Clearly a map is a quasi-similarity if and only if it is biLipschitz. The point here is that often there is control on $K$ but not on $C$. In this case, the notion of quasi-similarity provides more information about the distortion.

We say that a Carnot group $N$ is quasisymmetrically rigid if every $\eta$-quasisymmetric map from $N$ to $N$ is a $(K,C)$-quasi-similarity, where $N$ is equipped with a Carnot metric and $K$ is a constant depending only on $\eta$. See Section 2.3 for the definition of quasisymmetric map. A Carnot algebra is called quasisymmetrically rigid if the corresponding Carnot group is quasisymmetrically rigid. The main result of this paper says that most non-rigid Carnot groups are quasisymmetrically rigid. We next describe those exceptional non-rigid Carnot groups.

The first class of exceptional non-rigid Carnot groups are suitable quotients of direct products of the same Heisenberg group.

Let $m,n \geq 1$ be integers. Let $\tilde{\mathcal{N}}$ be the direct sum of $n$ copies of the $m$-th Heisenberg algebra $\mathcal{H}_C^m$. In other words, $\tilde{\mathcal{N}} = \tilde{\mathcal{H}}_1 \oplus \cdots \oplus \tilde{\mathcal{H}}_n$, where each $\tilde{\mathcal{H}}_j = \mathcal{H}_C^m$. Let $\tilde{V}_{1,j}$ and $\tilde{L}_j$ respectively be the first and second layers of $\tilde{\mathcal{H}}_j$. Let $\tilde{V}_1$ and $\tilde{V}_2$ respectively be the first and second layers of $\tilde{\mathcal{N}}$. Then $\tilde{V}_1 = \tilde{V}_{1,1} \oplus \cdots \oplus \tilde{V}_{1,n}$ and $\tilde{V}_2 = \tilde{L}_1 \oplus \cdots \oplus \tilde{L}_n$. Since $\tilde{V}_2$ is central in $\tilde{\mathcal{N}}$, every linear subspace $V \subset \tilde{V}_2$ is an ideal of $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{N}}/V = \tilde{V}_1 \oplus (\tilde{V}_2/V)$ is a Carnot algebra. Let $V \subset \tilde{V}_2$ be a linear subspace and $G_2 \subset GL(\tilde{V}_2)$ be a group of linear transformations of $\tilde{V}_2$. We call $\mathcal{N} := \tilde{\mathcal{N}}/V$ a Heisenberg product algebra if the following conditions are satisfied:

1. $\tilde{L}_j \cap V = \{0\}$ for all $j$;
2. $(V + \tilde{L}_i) \cap (V + \tilde{L}_j) = V$ for all $i \neq j$;
3. $V$ is $G_2$-invariant; that is, $g_2(V) = V$ for all $g_2 \in G_2$;
4. $G_2$ permutes the set $\{\tilde{L}_1, \cdots, \tilde{L}_n\}$ and acts transitively on the set.

A Carnot group is called a Heisenberg product group if its Lie algebra is a Heisenberg product algebra.

The second class of exceptional non-rigid Carnot groups are suitable quotients of direct products of the same complex Heisenberg group.

Recall that, the $m$-th ($m \geq 1$) complex Heisenberg algebra $\mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C}$ is a complex Lie algebra of dimension $2m + 1$. If $e_1, \cdots, e_{2m}, \eta$ are the standard basis of $\mathbb{C}^{2m} \oplus \mathbb{C}$, then the only non-trivial bracket relations (over $\mathbb{C}$) are $[e_i, e_{m+i}] = \eta$, $1 \leq i \leq m$.

Let $m,n \geq 1$ be integers. Let $\tilde{\mathcal{N}}$ be the direct sum of $n$ copies of the $m$-th complex Heisenberg algebra $\mathcal{H}_C^m$. In other words, $\tilde{\mathcal{N}} = \tilde{\mathcal{H}}_1 \oplus \cdots \oplus \tilde{\mathcal{H}}_n$, where each $\tilde{\mathcal{H}}_j = \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C}$. Let $\tilde{V}_{1,j}$ and $\tilde{L}_j$ respectively be the first and second layers of $\tilde{\mathcal{H}}_j$. We fix a graded isomorphism $f_j : \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \to \tilde{\mathcal{H}}_j$. Then $f_j|_{\{0\} \oplus \mathbb{C}} : \mathbb{C} \to \tilde{L}_j$ is a real linear isomorphism. Let $\tilde{V}_1$ and $\tilde{V}_2$ respectively be the first and second layers of $\tilde{\mathcal{N}}$. Then $\tilde{V}_1 = \tilde{V}_{1,1} \oplus \cdots \oplus \tilde{V}_{1,n}$ and $\tilde{V}_2 = \tilde{L}_1 \oplus \cdots \oplus \tilde{L}_n$. Since $\tilde{V}_2$ is central in $\tilde{\mathcal{N}}$, every linear subspace $V \subset \tilde{V}_2$ is an ideal of $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{N}}/V = \tilde{V}_1 \oplus (\tilde{V}_2/V)$ is a Carnot algebra. Let $V \subset \tilde{V}_2$ be a real linear subspace and $G_2 \subset GL(\tilde{V}_2)$ be a group of real linear transformations of $\tilde{V}_2$. We call $\mathcal{N} := \tilde{\mathcal{N}}/V$ a complex Heisenberg product algebra if the following conditions are satisfied:

1. $\tilde{L}_j \cap V = \{0\}$ for all $j$;
2. $V$ is $G_2$-invariant; that is, $g_2(V) = V$ for all $g_2 \in G_2$;
3. $G_2$ permutes the set $\{\tilde{L}_1, \cdots, \tilde{L}_n\}$ and acts transitively on the set;
4. For each $g_2 \in G_2$ and every $j$, if $g_2(\tilde{L}_j) = \tilde{L}_{\sigma(j)}$ for some $1 \leq \sigma(j) \leq n$, then there is
some $0 \neq a_j \in \mathbb{C}$ such that the map $g_{2,j} := f_{\sigma(j)}^{-1} \circ g_2 \circ f_j : \mathbb{C} \to \mathbb{C}$ has exactly one of the two forms $g_{2,j}(z) = a_j z$ or $g_{2,j}(z) = a_j \bar{z}$, where $\bar{z}$ is the complex conjugate of $z$. In other words, $g_{2,j}$ is a similarity.

A Carnot group is called a complex Heisenberg product group if its Lie algebra is a complex Heisenberg product algebra.

Here is the main result of the paper:

**Theorem 1.1.** Let $N$ be a non-rigid Carnot group. If $N$ is not one of the following three classes of groups, then it is quasisymmetrically rigid:

1. Euclidean groups;
2. Heisenberg product groups;
3. complex Heisenberg product groups.

It is known that there exist non-biLipschitz quasisymmetric maps on Euclidean groups [GV] and Heisenberg groups [B]. It is an open question whether the other Heisenberg product groups and complex Heisenberg product groups are quasisymmetrically rigid. Two prominent examples among these groups are complex Heisenberg groups and the direct product of at least two copies of the same Heisenberg group.

The results in this paper extend previous results by the author for reducible Carnot groups [X3], model Filiform groups [X1] and 2-step Carnot groups with reducible first layer [X2].

The first rigidity theorem about quasiconformal maps of Carnot groups is due to Pansu. He proved that ([P]) every quasiconformal map of the quarternionic Heisenberg group is a composition of left translations and graded automorphisms. In particular, the space of quasiconformal maps is finite dimensional. Our results are of somehow different nature. For example, Proposition 6.3 says that the space of biLipschitz maps of Carnot groups of rank one is infinite dimensional. Our main result implies that most of these groups are quasisymmetrically rigid.

The results in this paper have implications for the large scale geometry of negatively curved homogeneous manifolds. Each Carnot group arises as the (one point complement of) ideal boundary of some negatively curved homogeneous manifold [H]. Furthermore, each quasiisometry of the negatively curved homogeneous manifold associated to a quasisymmetrically rigid Carnot group is a rough isometry, that is, it must preserve the distance up to an additive constant.

In Section 2 we collect definitions and results that shall be needed later. In Section 3 we show that high step non-rigid Carnot algebras have reducible first layer. In Section 4 we give a characterization of non-rigid Carnot algebras with irreducible first layer. In Section 5 we prove the main result (Theorem 1.1) of the paper. Finally in Section 6 we construct an infinite dimensional space of biLipschitz maps on rank one Carnot groups.

**Acknowledgment.** This work was initiated while the author was attending the workshop “Interactions between analysis and geometry” at IPAM, University of California at Los Angeles from March to June 2013. I would like to thank IPAM for financial support, excellent working conditions and conducive atmosphere. I also would like to thank David Freeman and Tullia Dymarz for discussions about Carnot groups.
2 Preliminaries

In this Section we collect definitions and results that shall be needed later.

2.1 Carnot groups and Carnot algebras

A Carnot Lie algebra is a finite dimensional Lie algebra \( \mathcal{G} \) together with a direct sum decomposition \( \mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r \) of non-trivial vector subspaces such that \( [V_i, V_j] = V_{i+j} \) for all \( 1 \leq i \leq r \), where we set \( V_{r+1} = \{0\} \). The integer \( r \) is called the degree of nilpotency of \( \mathcal{G} \). Every Carnot algebra \( \mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r \) admits a one-parameter family of automorphisms \( \lambda_t : \mathcal{G} \to \mathcal{G} \), \( t \in (0, \infty) \), where \( \lambda_t(x) = t^i x \) for \( x \in V_i \). Let \( \mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r \) and \( \mathcal{G}' = V'_1 \oplus V'_2 \oplus \cdots \oplus V'_r \) be two Carnot algebras. A Lie algebra homomorphism \( \phi : \mathcal{G} \to \mathcal{G}' \) is graded if \( \phi \) commutes with \( \lambda_t \) for all \( t > 0 \); that is, if \( \phi \circ \lambda_t = \lambda_t \circ \phi \). We observe that \( \phi(V_i) \subset V'_i \) for all \( 1 \leq i \leq r \).

A simply connected nilpotent Lie group is a Carnot group if its Lie algebra is a Carnot algebra. Let \( G \) be a Carnot group with Lie algebra \( \mathcal{G} = V_1 \oplus \cdots \oplus V_r \). The subspace \( V_1 \) defines a left invariant distribution \( H \mathcal{G} \subset T \mathcal{G} \) on \( G \). We fix a left invariant inner product on \( H \mathcal{G} \). An absolutely continuous curve \( \gamma \) in \( G \) whose velocity vector \( \gamma'(t) \) is contained in \( H_{\gamma(t)} \mathcal{G} \) for a.e. \( t \) is called a horizontal curve. By Chow’s theorem ([BR, Theorem 2.4]), any two points of \( G \) can be connected by horizontal curves. Let \( p, q \in G \), the Carnot metric \( d_c(p, q) \) between them is defined as the infimum of length of horizontal curves that join \( p \) and \( q \).

Since the inner product on \( H \mathcal{G} \) is left invariant, the Carnot metric on \( G \) is also left invariant. Different choices of inner product on \( H \mathcal{G} \) result in Carnot metrics that are biLipschitz equivalent. The Hausdorff dimension of \( G \) with respect to a Carnot metric is given by \( \sum_{i=1}^{r} i \cdot \dim(V_i) \).

Recall that, for a simply connected nilpotent Lie group \( G \) with Lie algebra \( \mathcal{G} \), the exponential map \( \exp : \mathcal{G} \to G \) is a diffeomorphism. Under this identification the Lebesgue measure on \( \mathcal{G} \) is a Haar measure on \( G \). Furthermore, the exponential map induces a one-to-one correspondence between Lie subalgebras of \( \mathcal{G} \) and connected Lie subgroups of \( G \).

It is often more convenient to work with homogeneous distances defined using norms than with Carnot metrics. Let \( \mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r \) be a Carnot algebra. Write \( x \in \mathcal{G} \) as \( x = x_1 + \cdots + x_r \) with \( x_i \in V_i \). Fix a norm \( \| \cdot \| \) on each layer. Define a norm \( || \cdot || \) on \( \mathcal{G} \) by:

\[
||x|| = \sum_{i=1}^{r} |x_i|^\frac{1}{2}.
\]

Now define a homogeneous distance on \( G = \mathcal{G} \) by: \( d(g, h) = ||(-g) * h|| \). In general, \( d \) is only a quasimetric. However, \( d \) and \( d_c \) are always biLipschitz equivalent. That is, there is a constant \( C \geq 1 \) such that \( d(p, q)/C \leq d_c(p, q) \leq C \cdot d(p, q) \) for all \( p, q \in G \). It is often possible to calculate or estimate \( d \) by using the BCH formula (see Subsection 2.2). Since we are only concerned with quasiconformal maps and biLipschitz maps, it does not matter whether we use \( d \) or \( d_c \).
2.2 The Baker-Campbell-Hausdorff formula

Let $G$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. One can then pull back the group operation from $G$ to get a group structure on $\mathfrak{g}$. This group structure can be described by the Baker-Campbell-Hausdorff formula (BCH formula in short), which expresses the product $X * Y$ ($X, Y \in \mathfrak{g}$) in terms of the iterated Lie brackets of $X$ and $Y$. The group operation in $G$ will be denoted by $\cdot$. The pull-back group operation $*$ on $\mathfrak{g}$ is defined as follows. For $X, Y \in \mathfrak{g}$, define

$$X * Y = \exp^{-1}(\exp X \cdot \exp Y).$$

Then the BCH formula ([CG], page 11) says

$$X * Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{p_i+q_i>0, 1 \leq i \leq n} \frac{(\sum_{i=1}^{n} (p_i + q_i))^{-1}}{p_1!q_1! \cdots p_n!q_n!} (\ad X)^{p_1}(\ad Y)^{q_1} \cdots (\ad X)^{p_n}(\ad Y)^{q_n-1} Y,$$

where $\ad A(B) = [A, B]$. If $q_n = 0$, the term in the sum is $\cdots (\ad X)^{p_n-1} X$; if $q_n > 1$ or if $q_n = 0$ and $p_n > 1$, then the term is zero. The first a few terms are well known,

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] - \frac{1}{48}[Y, [X, [X, Y]]] - \frac{1}{48}[X, [Y, [X, Y]]] + (\text{commutators in five or more terms}).$$

2.3 Quasisymmetric map

Here we recall the definitions of quasisymmetric maps and pointwise Lipschitz constant.

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. A homeomorphism $F : X \to Y$ between two metric spaces is $\eta$-quasisymmetric if for all distinct triples $x, y, z \in X$, we have

$$\frac{d(F(x), F(y))}{d(F(x), F(z))} \leq \eta \left( \frac{d(x, y)}{d(x, z)} \right).$$

If $F : X \to Y$ is an $\eta$-quasisymmetry, then $F^{-1} : Y \to X$ is an $\eta_1$-quasisymmetry, where $\eta_1(t) = (\eta^{-1}(t^{-1}))^{-1}$. See [V], Theorem 6.3. A homeomorphism between metric spaces is quasisymmetric if it is $\eta$-quasisymmetric for some $\eta$.

We remark that quasisymmetric homeomorphisms between general metric spaces are quasiconformal. In the case of Carnot groups (and more generally Loewner spaces), a homeomorphism is quasisymmetric if and only if it is quasiconformal, see [HK].

Let $F : X \to Y$ be a homeomorphism between two metric spaces. We define for every $x \in X$ and $r > 0$,

$$L_F(x, r) = \sup\{d(F(x), F(x')) : d(x, x') \leq r\},$$

$$l_F(x, r) = \inf\{d(F(x), F(x')) : d(x, x') \geq r\},$$

and set

$$L_F(x) = \limsup_{r \to 0} \frac{L_F(x, r)}{r}, \quad l_F(x) = \liminf_{r \to 0} \frac{l_F(x, r)}{r}. $$
Then
\[ L_{F^{-1}}(F(x)) = \frac{1}{l_F(x)} \quad \text{and} \quad l_{F^{-1}}(F(x)) = \frac{1}{L_F(x)} \] (2.1)
for any \( x \in X \). If \( F \) is an \( \eta \)-quasisymmetry, then
\[ L_F(x, r) \leq \eta(1/l_F(x, r)) \] (2.2)
for all \( x \in X \) and \( r > 0 \).

### 2.4 Previous results

Here we collect some results that will be used in this paper.

A \( C^2 \) diffeomorphism between domains of Carnot groups whose differential preserves the horizontal bundle is called a contact map. A Carnot group \( N \) is rigid if the space of \( C^2 \)-contact maps between domains of \( N \) is finite dimensional, otherwise, \( N \) is non-rigid. A Carnot algebra is called rigid if the corresponding Carnot group is rigid. Similarly a Carnot algebra is called non-rigid if the corresponding Carnot group is non-rigid.

Non-rigid Carnot groups (algebras) can be characterized in terms of rank. The rank of an element in a Lie algebra appeared in [O].

For an element \( x \in N \) in a Lie algebra, let \( \text{rank}(x) \) be the rank of the linear transformation \( \text{ad}(x) : N \rightarrow N, \text{ad}(x)(y) = [x, y] \). In other words, \( \text{rank}(x) \) is the dimension of the image of \( \text{ad}(x) \). It is easy to see that \( \text{rank}(x) = \text{rank}(F(x)) \) for any isomorphism \( F : N_1 \rightarrow N_2 \).

**Theorem 2.1.** ([OW], Theorem 1) Let \( N \) be a Carnot algebra. Then \( N \) is non-rigid if and only if there exists a nonzero vector \( X \) in the first layer of the complexification of \( N \) such that \( \text{rank}(X) \leq 1 \).

We say that a Carnot group \( N \) is quasisymmetrically rigid if every \( \eta \)-quasisymmetric map from \( N \) to \( N \) is a \((K,C)\)-quasi-similarity, where \( N \) is equipped with a Carnot metric and \( K \) is a constant depending only on \( \eta \). A Carnot algebra is quasisymmetrically rigid if the corresponding Carnot group is.

Let \( \text{Aut}_g(N) \) be the group of graded isomorphisms of a Carnot algebra \( N \). We say \( V_1 \) is reducible (or the first layer of \( N \) is reducible) if there is a non-trivial proper linear subspace \( W \subset V_1 \) such that \( A(W) = W \) for every \( A \in \text{Aut}_g(N) \).

**Theorem 2.2.** ([X2], Theorem 1.1) Let \( N \) be a 2-step Carnot algebra. If the first layer of \( N \) is reducible, then \( N \) is quasisymmetrically rigid.

**Theorem 2.3.** ([X2], Theorem 1.2) Suppose \( W_1 \subset V_1 \) is a non-trivial proper subspace that is invariant under the action of \( \text{Aut}_g(N) \). If there is some \( X \in V_1 \setminus W_1 \) such that \( [X, W_1] \subset [W_1, W_1] \), then \( N \) is quasisymmetrically rigid.

The following result is very useful for induction argument:

**Theorem 2.4.** ([X2], Theorem 1.3) Suppose \( W_1 \subset V_1 \) is a non-trivial proper subspace that is invariant under the action of \( \text{Aut}_g(N) \). If the Lie subalgebra \(<W_1>\) generated by \( W_1 \) is quasisymmetrically rigid, then so is \( N \).
We shall also need the following result (Section 5.2).

**Proposition 2.5.** ([X3], Proposition 3.4) Let \( G \) and \( G' \) be two Carnot groups with Lie algebras \( \mathcal{G} = V_1 \oplus \cdots \oplus V_m \) and \( \mathcal{G}' = V'_1 \oplus \cdots \oplus V'_n \) respectively. Let \( W_1 \subset V_1, W'_1 \subset V'_1 \) be linear subspaces. Denote by \( W \subset \mathcal{G} \) and \( W' \subset \mathcal{G}' \) respectively the Lie subalgebras generated by \( W_1 \) and \( W'_1 \). Let \( W \subset G \) and \( W' \subset G' \) respectively be the connected Lie subgroups of \( G \) and \( G' \) corresponding to \( W \) and \( W' \). Let \( F : G \to G' \) be a quasisymmetric homeomorphism. If \( dF(x)(W_1) \subset W'_1 \) for a.e. \( x \in G \), then \( F \) sends left cosets of \( W \) into left cosets of \( W' \).

### 3 High step non-rigid Carnot algebras have reducible first layer

The goal of this section is to show that if \( N \) is non-rigid and has step \( r \geq 3 \), then the first layer is reducible.

Let \( N = V_1 \oplus \cdots \oplus V_r \) be a Carnot algebra over a field \( K \). We are only interested in the cases \( K = \mathbb{R} \) and \( K = \mathbb{C} \). Define

\[
    r_{1,K}(N) = \min \{ \text{rank}(x) : 0 \neq x \in V_1 \}.
\]

When \( K = \mathbb{R} \), we simply write \( r_1(N) \) for \( r_{1,\mathbb{R}}(N) \) and call it the rank of \( N \) and of the corresponding Carnot group \( N \). Let \( W_1 \subset V_1 \) be the linear subspace of \( V_1 \) spanned by elements \( x \in V_1 \) satisfying \( \text{rank}(x) = r_1(N) \). Since graded isomorphisms preserve the first layer \( V_1 \), we see that \( W_1 \) is a non-trivial subspace of \( V_1 \) invariant under the action of Aut\(_N\)(\( N \)). The question is when \( W_1 \) is a proper subspace of \( V_1 \).

**Lemma 3.1.** Let \( N \) be a non-abelian Carnot algebra. If \( r_1(N) = 0 \), then \( W_1 \) is a proper subspace of \( V_1 \).

**Proof.** Let \( X, Y \in V_1 \) with \( \text{rank}(X) = \text{rank}(Y) = 0 \), and \( a, b \in \mathbb{R} \). Then \( [X, N] = [Y, N] = 0 \). Hence \( [aX + bY, N] = a[X, N] + b[Y, N] = 0 \). It follows that \( W_1 \) is the set of elements in \( V_1 \) with rank 0. Since \( N \) is non-abelian, \( W_1 \neq V_1 \).

**Lemma 3.2.** For any Carnot Lie algebra \( N \), \( r_1(N) = 0 \) if and only if \( N \) can be written as a direct sum of an Euclidean algebra and another Carnot algebra.

**Proof.** First suppose \( N = \mathbb{R}^k \oplus N' = (\mathbb{R}^k \oplus V'_1) \oplus V'_2 \oplus \cdots \oplus V'_r \) is a direct sum of an Euclidean algebra and another Carnot algebra. Clearly every \( X \in \mathbb{R}^k \setminus \{0\} \subset V_1 = \mathbb{R}^k \oplus V'_1 \) has rank 0.

Conversely, assume there exists a nonzero \( X \in V_1 \) with \( \text{rank}(X) = 0 \). Let \( V'_1 \subset V_1 \) be a codimension 1 subspace complementary to \( \mathbb{R}X \). Set \( N' = V'_1 \oplus V_2 \oplus \cdots \oplus V_r \). Then it is easy to check that \( [V'_1, V'_1] = V_2 \) and \( [V'_i, V_i] = V_{i+1} \) for \( 2 \leq i \leq r \), where \( V_{r+1} = 0 \). It follows that \( N' \) is also a Carnot algebra. Now it is easy to check that the map \( \mathbb{R} \oplus N' \to N \), \((a, Y) \to aX + Y\), is a graded isomorphism of Carnot algebras.
Lemma 3.3. Suppose \( r_1(\mathcal{N}) = 1 \) and \( W_1 = V_1 \). Then \( \mathcal{N} \) is a 2-step Carnot algebra.

Proof. The assumptions imply that there is a vector space basis \( X_1, \ldots, X_m \) of \( V_1 \) satisfying \( \text{rank}(X_i) = 1 \) for all \( 1 \leq i \leq m \). Since \( \mathcal{N} \) is generated by \( V_1 \), we must have \([X_i, V_1]\) \( \neq 0 \). Hence for each \( i \) there is some \( j_i \) such that \( Y_i := [X_i, X_{j_i}] \neq 0 \). Notice that \( Y_i \subset V_2 \). Hence \( V_2 \neq 0 \). Since \( \text{rank}(X_i) = 1 \), we have \([X_i, \mathcal{N}] = \mathbb{R} Y_i \). Since \([X_i, V_j] \subset V_{j+1} \), we have \([X_i, V_2] = 0 \). Since \( X_1, \ldots, X_m \) form a basis of \( V_1 \), we have \( V_3 = [V_1, V_2] = 0 \). So \( \mathcal{N} \) is a 2-step Carnot algebra.

\( \square \)

Corollary 3.4. Let \( \mathcal{N} \) be a \( r \)-step Carnot algebra with \( r \geq 3 \). If \( r_1(\mathcal{N}) \leq 1 \), then \( W_1 \neq V_1 \).

Proof. If \( r_1(\mathcal{N}) = 0 \), the claim follows from Lemma 3.1. So we assume \( r_1(\mathcal{N}) = 1 \). Since \( \mathcal{N} \) is \( r \)-step with \( r \geq 3 \), the Corollary now follows from Lemma 3.3.

Let \( \mathcal{N}_C = \mathcal{N} \otimes \mathbb{C} = (V_1 \otimes \mathbb{C}) \oplus \cdots \oplus (V_r \otimes \mathbb{C}) \) be the complexification of \( \mathcal{N} \). Let \( r_{1,C}(\mathcal{N}) = \min \{ \text{rank}(X) \mid \exists X \in V_1 \otimes \mathbb{C} \} \) and \( W_{1,C} \subset V_1 \otimes \mathbb{C} \) be the complex linear subspace of \( V_1 \otimes \mathbb{C} \) spanned by elements \( X \notin V_1 \otimes \mathbb{C} \) with \( \text{rank}(X) = r_{1,C}(\mathcal{N}) \).

Lemma 3.5. Suppose \( r_1(\mathcal{N}) \geq 2 \). Then \( r_{1,C}(\mathcal{N}) \geq 1 \). If \( r_{1,C}(\mathcal{N}) = 1 \) and \( W_{1,C} = V_1 \otimes \mathbb{C} \), then \( \mathcal{N} \) is a 2-step Carnot algebra.

Proof. Suppose there is some \( X = X_1 + iX_2 \in V_1 \otimes \mathbb{C} \) with \( \text{rank}(X) = 0 \), where \( X_1, X_2 \in V_1 \). Then \([X, \mathcal{N}] \subset [X, \mathcal{N}_C] = 0 \). It follows that \([X_1, \mathcal{N}] = [X_2, \mathcal{N}] \). This implies there is a nonzero element \((X_1 \lor X_2)\) in \( V_1 \) with \( \text{rank} 0 \), contradicting the assumption \( r_1(\mathcal{N}) \geq 2 \). Hence \( r_{1,C}(\mathcal{N}) \geq 1 \).

Now suppose \( r_{1,C}(\mathcal{N}) = 1 \) and \( W_{1,C} = V_1 \otimes \mathbb{C} \). Then the proof of Lemma 3.3 shows \([V_1 \otimes \mathbb{C}, V_2 \otimes \mathbb{C}] = 0 \). Hence \([V_1 \otimes \mathbb{C}, V_2 \otimes \mathbb{C}] \subset [V_1, V_2] = 0 \), which implies \( \mathcal{N} \) is 2-step.

\( \square \)

Lemma 3.6. If \( r_{1,C}(\mathcal{N}) \leq 1 \), then \( r_1(\mathcal{N}) \leq 2 \).

Proof. In the proof of Lemma 3.5 we already showed that if \( r_{1,C}(\mathcal{N}) = 0 \), then \( r_1(\mathcal{N}) = 0 \). So we shall assume \( r_{1,C}(\mathcal{N}) = 1 \). Let \( 0 \neq X = X_1 + iX_2 \in V_1 \otimes \mathbb{C} \) \((X_1, X_2 \in V_1)\) with \( \text{rank}(X) = 1 \). By multiplying \( i \) if necessary we may assume \( X_1 \neq 0 \). We must have \([X, V_1] \neq 0 \), otherwise \([X, \mathcal{N} \otimes \mathbb{C}] = 0 \). So there is some \( Y_0 \in V_1 \) such that \([X, Y_0] \neq 0 \). Since \( r_{1,C}(\mathcal{N}) = 1 \), we have \([X, \mathcal{N} \otimes \mathbb{C}] = \mathbb{C}[X, Y_0] \). In particular, for any \( Y \in \mathcal{N} \), there are \( a, b \in \mathbb{R} \) such that

\[
[X_1 + iX_2, Y] = [X, Y] = (a + ib)[X, Y_0] = (a + ib)[X_1 + iX_2, Y_0].
\]

Comparing the real parts of both sides, we obtain \([X_1, Y] = a[X_1, Y_0] - b[X_2, Y_0] \). This implies \([X_1, \mathcal{N}] \subset \mathbb{R}[X_1, Y_0] + \mathbb{R}[X_2, Y_0] \). Hence \( \text{rank}(X_1) \leq 2 \).

\( \square \)

Corollary 3.7. Let \( \tilde{W}_1 = \{ X \in V_1 \mid \text{there exists } Y \in V_1 \text{ such that } X + iY \in W_{1,C} \} \). If \( r_1(\mathcal{N}) \geq 2 \), \( r_{1,C}(\mathcal{N}) = 1 \) and \( \mathcal{N} \) is \( r \)-step with \( r \geq 3 \), then \( \tilde{W}_1 \neq V_1 \).
Proof. By Lemma 3.5 and our assumption that \( r \geq 3 \), we have \( W_{1,\mathbb{C}} \neq V_1 \otimes \mathbb{C} \). Let \( X_1, \ldots, X_m \) be a basis for the complex vector space \( W_{1,\mathbb{C}} \) satisfying \( \text{rank}(X_j) = 1 \). We write \( X_j = Y_j + iZ_j \) with \( Y_j, Z_j \in V_1 \). Then it is easy to check that as a real vector space \( \hat{W}_1 \) is spanned by \( Y_1, Z_1, \ldots, Y_m, Z_m \). Suppose \( \hat{W}_1 = V_1 \). Then \( Y_1, Z_1, \ldots, Y_m, Z_m \) span \( V_1 \). As seen in the proof of Lemma 3.3, we have \( \langle X_j, V_2 \otimes \mathbb{C} \rangle = 0 \). In particular, \( \langle Y_j + iZ_j, V_2 \rangle = 0 \). It follows that \( \langle Y_j, V_2 \rangle = 0 = \langle Z_j, V_2 \rangle \). Since \( Y_1, Z_1, \ldots, Y_m, Z_m \) span \( V_1 \), we have \( V_3 = \langle V_1, V_2 \rangle = 0 \), contradicting our assumption that \( r \geq 3 \). Hence \( \hat{W}_1 \neq V_1 \).

\[ \square \]

Proposition 3.8. If a Carnot algebra \( \mathcal{N} \) has step \( r \geq 3 \) and is non-rigid, then \( V_1 \) is reducible.

Proof. First of all, \( \text{rank}(L(X)) = \text{rank}(X) \) for any Lie algebra isomorphism \( L : \mathcal{N}_1 \to \mathcal{N}_2 \) and any \( X \in \mathcal{N}_1 \). A graded isomorphism between Carnot algebras also preserves the first layer. Hence for any \( L \in \text{Aut}_g(\mathcal{N}) \) and any \( X \in V_1 \), we have \( L(X) \in V_1 \) and \( \text{rank}(L(X)) = \text{rank}(X) \). It follows that \( L(W_1) = W_1 \). Similarly, \( L(W_{1,\mathbb{C}}) = W_{1,\mathbb{C}} \) and \( L(\hat{W}_1) = \hat{W}_1 \). Hence \( W_1 \) and \( \hat{W}_1 \) are non-trivial linear subspaces of \( V_1 \) that are invariant under the action of \( \text{Aut}_g(\mathcal{N}) \). Next we shall decide whether they are proper subspaces of \( V_1 \).

Since \( \mathcal{N} \) is non-rigid, Theorem 2.1 implies \( r_{1,\mathbb{C}}(\mathcal{N}) \leq 1 \). If \( r_1(\mathcal{N}) \leq 1 \), Corollary 3.4 implies \( W_1 \) is proper. Now assume \( r_1(\mathcal{N}) \geq 2 \). Then Lemma 3.5 and Corollary 3.7 imply \( \hat{W}_1 \neq V_1 \).

\[ \square \]

4 Non-rigid Carnot algebras with irreducible first layer

In this Section we give a characterization of non-rigid Carnot algebras with irreducible first layer.

Let \( \mathcal{N} = V_1 \oplus \cdots \oplus V_r \) be a Carnot algebra over a field \( K \). We are only interested in the cases \( K = \mathbb{R} \) and \( K = \mathbb{C} \). Let \( A_1 = \{ v \in V_1 : \text{rank}(v) = 1 \} \) be the set of elements in \( V_1 \) with rank 1. Notice that if \( v \in A_1 \), then \( [v, \mathcal{N}] = KZ \) for some \( 0 \neq Z \in V_2 \). We define an equivalence relation on \( A_1 \) as follows: \( v \sim v' \) if \( [v, \mathcal{N}] = [v', \mathcal{N}] \). It is easy to check that this is indeed an equivalence relation.

Lemma 4.1. Suppose \( r_{1,K}(\mathcal{N}) = 1 \). Let \( A \) be an equivalence class. Then \( A \cup \{ 0 \} \) is a linear subspace of \( V_1 \).

Proof. \( A \cup \{ 0 \} \) is clearly closed under scalar multiplication. We need to show that it is also closed under addition. Let \( X_1, X_2 \in A \cup \{ 0 \} \). We may assume \( X_1, X_2, X_1 + X_2 \neq 0 \). Then \( X_1 \sim X_2 \) and so there is some \( 0 \neq Z \in V_2 \) such that \( [X_1, \mathcal{N}] = KZ = [X_2, \mathcal{N}] \). So \( [X_1 + X_2, \mathcal{N}] \subset KZ \) and \( \text{rank}(X_1 + X_2) \leq 1 \). Since \( X_1 + X_2 \neq 0 \) and \( r_{1,K}(\mathcal{N}) = 1 \), we must have \( \text{rank}(X_1 + X_2) = 1 \) and \( [X_1 + X_2, \mathcal{N}] = KZ \). Hence \( X_1 + X_2 \in A \).

\[ \square \]
Lemma 4.2. Let $A_1, A_2$ be two distinct equivalence classes. Then $[X_1, X_2] = 0$ for any $X_1 \in A_1, X_2 \in A_2$.

Proof. There are $0 \neq Z_1, Z_2 \in V_2$ such that $KZ_1 \neq KZ_2$ and $[X_1, \mathcal{N}] = KZ_1$ and $[X_2, \mathcal{N}] = KZ_2$. So $[X_1, X_2] \in KZ_1 \cap KZ_2 = \{0\}$. 

We call a subset of the form $\tilde{A} := A \cup \{0\}$ an extended equivalence class, where $A$ is an equivalence class.

Example Let $\mathcal{N} = V_1 \oplus V_2$ be a 2-step Carnot algebra defined as follows. The first layer $V_1$ has a vector space basis $X_1, X_2, Y$ and the second layer $V_2$ has a vector space basis $Z_1, Z_2$. The only non-trivial bracket relations are $[X_1, Y] = Z_1, [X_2, Y] = Z_2$. It is easy to check that $r_1(\mathcal{N}) = 1$, and that elements of rank 1 in $V_1$ are exactly the nonzero elements in $\mathbb{R}X_1 \oplus \mathbb{R}X_2$ and $[a_1X_1 + a_2X_2, \mathcal{N}] = \mathbb{R}(a_1Z_1 + a_2Z_2)$. It follows that the extended equivalence classes are exactly the 1-dimensional subspaces of $\mathbb{R}X_1 \oplus \mathbb{R}X_2$. Consider linear subspaces of $V_1$ that consist of only rank 1 elements (and 0). These subspaces are partially ordered by inclusion. The above example shows that, in general, the extended equivalence classes are not maximal with respect to inclusion and their span is not a direct sum of all the extended equivalence classes. This picture changes when $V_1$ is spanned by rank 1 elements.

Lemma 4.3. Let $K = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{N}$ be a Carnot algebra over $K$. Suppose $r_{1,K}(\mathcal{N}) = 1$ and $V_1$ is spanned by rank 1 elements. Then

(1) $V_1$ is the direct sum of extended equivalence classes;
(2) Each extended equivalence class is a maximal linear subspace (over $K$) of $V_1$ consisting of (0 and) rank 1 elements;
(3) For each extended equivalence class $\tilde{A}$, the Lie subalgebra $< \tilde{A} >$ is isomorphic to a Heisenberg algebra over $K$.

Proof. (1) and (2). First we notice that two distinct extended equivalence classes intersect trivially. Let $A_1$ and $A_2$ be two distinct extended equivalence classes. Since $A_1$ and $A_2$ are distinct equivalence classes, we have $A_1 \cap A_2 = \emptyset$. Hence $A_1 \cap A_2 = \{0\}$.

Since every rank 1 element lies in an equivalence class, the assumption implies that $V_1$ is spanned by extended equivalence classes. Hence there are extended equivalence classes $\tilde{A}_1, \ldots, \tilde{A}_m$ such that $V_1 = \tilde{A}_1 + \cdots + \tilde{A}_m$. We may assume

$$\tilde{A}_1 + \cdots + \tilde{A}_j \neq \tilde{A}_1 + \cdots + \tilde{A}_{j+1}$$

for $1 \leq j \leq m - 1$. Now fix a subspace $B_{j+1} \subset \tilde{A}_{j+1}$ such that (1) $\tilde{A}_1 + \cdots + \tilde{A}_{j+1} = \tilde{A}_1 + \cdots + \tilde{A}_j + B_{j+1}$ and (2) $\tilde{A}_1 + \cdots + \tilde{A}_j \cap B_{j+1} = \{0\}$. The first paragraph implies $B_2 = \tilde{A}_2$. Set $B_1 = \tilde{A}_1$. Then

$$V_1 = B_1 \oplus B_2 \oplus B_3 \oplus \cdots \oplus B_m. \quad (4.1)$$

Since $B_j \subset \tilde{A}_j$, Lemma 4.2 implies $[B_i; B_j] = 0$ for $i \neq j$. Since $r_{1,K}(\mathcal{N}) = 1$, for any $0 \neq X \in B_j$, there exists some $Y \in B_j$ with $[X, Y] \neq 0$. 

We claim that \( \text{rank}(X) \geq 2 \) for any \( X \in V_1 \setminus (\cup_j B_j) \). Note that (2) follows from the claim. Now we prove the claim. Write \( X = X_{j_1} + \cdots + X_{j_k} \) with \( k \geq 2, 1 \leq j_1 < \cdots < j_k \leq m, 0 \neq X_{j_i} \in B_{j_i} \). For each \( 1 \leq j \leq m \), let \( 0 \neq Z_j \in V_2 \) be such that \([ Y, N ] = KZ_j \) for every \( Y \in A_j \). By the preceding paragraph, there exists some \( Y_{j_i} \in B_{j_i} \) with \([ X_{j_i}, Y_{j_i} ] = a_{j_i}Z_{j_i} \) for some \( a_{j_i} \neq 0 \). Now we have \([ X, Y_{j_1} ] = [ X_{j_1}, Y_{j_1} ] = a_{j_1}Z_{j_1} \) and \([ X, Y_{j_2} ] = [ X_{j_2}, Y_{j_2} ] = a_{j_2}Z_{j_2} \). Since \( KZ_{j_1} \neq KZ_{j_2} \), we see that \( \text{rank}(X) \geq 2 \). This in particular implies \( A_3 \cap (A_1 \oplus A_2) = \{0\} \) because the elements in \( A_3 \) have rank 1 and the elements in \((A_1 \oplus A_2) \setminus (A_1 \cup A_2)\) have rank (at least) 2. So we actually have \( B_3 = \tilde{A}_3 \). By induction we obtain \( B_j = \tilde{A}_j \) for all \( j \). Hence (1) follows from (4.1).

(3) For any \( X \in A_j \), there is some \( Y \in \tilde{A}_j \) such that \([ X, Y ] = aZ_j \) for some \( a \neq 0 \). Now (3) follows from the following Lemma.

\[ \square \]

**Lemma 4.4.** Let \( K \) be a field and \( N = V_1 \oplus V_2 \) be a 2-step Carnot algebra over \( K \) with \( \dim(V_2) = 1 \). Suppose for any \( 0 \neq X \in V_1 \) there is some \( Y \in V_1 \) such that \([ X, Y ] \neq 0 \). Then \( N \) is a Heisenberg algebra over \( K \).

**Proof.** Let \( Z \in V_2 \) be a nonzero element. Fix any \( X_1 \in V_1 \setminus \{0\} \). The assumption implies that there is some \( Y_1 \in V_1 \) with \([ X_1, Y_1 ] = Z \). Since \( \dim(V_2) = 1 \) and \([ V_1, V_1 ] \subset V_2 \), we see that \( \ker(ad\ X_1) \cap \ker(ad\ Y_1) \) has codimension 2 in \( V_1 \) and

\[ V_1 = KX_1 \oplus KY_1 \oplus (\ker(ad\ X_1) \cap \ker(ad\ Y_1)). \]

If \( \ker(ad\ X_1) \cap \ker(ad\ Y_1) = \{0\} \), then \( N \) is isomorphic to the first Heisenberg algebra over \( K \) and we are done. Assume \( \ker(ad\ X_1) \cap \ker(ad\ Y_1) \neq \{0\} \). In this case, it is easy to check that the subalgebra \( N_1 := \langle (\ker(ad\ X_1) \cap \ker(ad\ Y_1)) \rangle \) satisfies the assumption in the Lemma. By induction we may assume \( N_1 \) is the \( m \)-th Heisenberg algebra over \( K \) for some \( m \geq 1 \). Hence there are \( X_j, Y_j \in \ker(ad\ X_1) \cap \ker(ad\ Y_1) \) for \( 2 \leq j \leq m+1 \) such that

1. \([ X_j, Y_j ] = Z \) for all \( 2 \leq j \leq m+1 \) and \([ X_i, Y_j ] = 0, [ X_i, Y_j ] = 0 \) for \( i \neq j \);
2. \( \ker(ad\ X_1) \cap \ker(ad\ Y_1) = KX_1 \oplus KY_1 \oplus \cdots \oplus KX_{m+1} \oplus KY_{m+1} \).

Now it is easy to see that \( N \) is the \((m+1)\)-th Heisenberg algebra over \( K \).

\[ \square \]

Recall that \( r_{1,C}(N) \) and \( W_{1,C} \) are defined before Lemma 3.5.

**Lemma 4.5.** Let \( N \) be a Carnot algebra. Suppose \( r_1(N) \geq 2 \) and \( r_{1,C}(N) \leq 1 \). Assume the action of \( \text{Aut}_g(N) \) on \( V_1 \) is irreducible. Then \( V_1 \) can be written as a direct sum of vector subspaces \( V_1 = U_1 \oplus \cdots \oplus U_n \) such that \([ U_j, U_{j'} ] = 0 \) for \( j \neq j' \), and for each \( j, < U_j > \) is isomorphic to a complex Heisenberg algebra.

**Proof.** Lemma 3.5 implies \( r_{1,C}(N) = 1 \). We claim that \( W_{1,C} = V_1 \otimes \mathbb{C} \) holds. Suppose \( W_{1,C} \neq V_1 \otimes \mathbb{C} \). Define projections \( \pi_1 : W_{1,C} \to V_1 \) and \( \pi_2 : W_{1,C} \to V_1 \) by \( \pi_1(X_1 + iX_2) = X_1 \) and \( \pi_2(X_1 + iX_2) = X_2 \) respectively. Clearly \( \pi_1 \) and \( \pi_2 \) commute with the action of \( \text{Aut}_g(N) \). Since \( W_{1,C} \) is invariant under the action of \( \text{Aut}_g(N) \), we see that \( \pi_1(W_{1,C}) \) is a subspace of \( V_1 \) invariant under the action of \( \text{Aut}_g(N) \). Notice that \( \pi_1(W_{1,C}) \) is non-trivial since \( W_{1,C} \) is non-trivial and is a complex subspace of \( V_1 \otimes \mathbb{C} \) (so if \( X_1 + iX_2 \in W_{1,C} \), then
\(i(X_1 + iX_2) \in W_{1,C}\). Since our assumption the action of \(\text{Aut}_g(N)\) on \(V_1\) is irreducible, we must have \(\pi_1(W_{1,C}) = V_1\). Similarly the kernel \(\ker(\pi_1) \subset iV_1\) is also invariant under the action of \(\text{Aut}_g(N)\). Hence \(\ker(\pi_1) = \{0\}\) or \(iV_1\). Since we assumed \(W_{1,C} \neq V_1 \otimes C\), we must have \(\ker(\pi_1) = 0\). A similar argument shows that \(\pi_2 : W_{1,C} \to V_1\) is onto with trivial kernel. It follows that there is a (real) linear isomorphism \(J : V_1 \to V_1\) such that \(W_{1,C} = \{(X + ij(X) : X \in V_1)\}\). Notice that, if \(X = X_1 + iX_2 \in V_1 \otimes C\) has rank 1, then its complex conjugate \(\overline{X} = X_1 - iX_2\) also has rank 1. Since by definition \(W_{1,C}\) is spanned by rank one elements, \(\overline{W}_{1,C} = \{X_1 - iX_2 : X_1 + iX_2 \in W_{1,C}\}\) is also spanned by rank 1 elements. It follows that \(\overline{W}_{1,C} \subset W_{1,C}\). Consequently, \(W_{1,C} = V_1 \otimes C\), contradicting the facts that \(\pi_1\) and \(\pi_2\) have trivial kernels. The contradiction implies \(W_{1,C} = V_1 \otimes C\).

Now we apply Lemma 4.3 to \(N_1 \otimes C\) and conclude that \(V_1 \otimes C\) is a direct sum of extended equivalence classes \(V_1 \otimes C = W_1 \oplus \cdots \oplus W_m\), and \(<W_j>\) is isomorphic to a complex Heisenberg algebra. Note that the conjugate \(\overline{W}_j\) of \(W_j\) is also an extended equivalence class, so it equals some \(W_k\). We claim that \(\overline{W}_j \cap W_j = \{0\}\). Suppose \(\overline{W}_j \cap W_j \neq \{0\}\). Fix any \(0 \neq X = X_1 + iX_2 \in \overline{W}_j \cap W_j\), where \(X_1, X_2 \in V_1\). Then \(\overline{X} = X_1 - iX_2 \in W_j\). Since \(W_j\) is a complex linear subspace of \(V_1 \otimes C\), we have \(X_1, X_2 \in W_j\). At least one of \(X_1, X_2\) is nonzero. We may assume \(X_1 \neq 0\). Since \(W_j\) is an extended equivalence class, every nonzero element in \(W_j\) has rank 1. Hence there is some \(Y \in V_1\) with \([X_1, Y] \neq 0\), otherwise \([X_1, N_1 \otimes C] = 0\). It follows that \([X_1, N_1 \otimes C] = C[X_1, Y]\). Therefore \([X_1, N] \subset C[X_1, Y] \cap N = \mathbb{R}[X_1, Y]\). This means \(X_1 \neq 0\) has rank at most 1, contradicting the assumption that \(r_1(N) \geq 2\). Hence \(\overline{W}_j \cap W_j = \{0\}\). Therefore \(W_j = W_k\) for some \(k \neq j\). In particular \(m = 2n\) for some \(n\). We may relabel the \(W_j\)'s so that \(W_{n+j} = \overline{W}_j\) for \(1 \leq j \leq n\).

Consider \(W_j \oplus \overline{W}_j\) (\(1 \leq j \leq n\)). Let \(U_j = \pi_1(W_j) \subset V_1\). Then \(W_j \oplus \overline{W}_j = U_j \oplus iU_j\). Hence \(\dim_{\mathbb{C}}(W_j \oplus \overline{W}_j) = \dim_{\mathbb{R}} U_j\). Notice that \(V_1 = U_1 \oplus \cdots \oplus U_n\). Since \(V_1 \otimes C = (W_1 \oplus \overline{W}_1) \oplus \cdots \oplus (W_n \oplus \overline{W}_n)\) and \(\dim_{\mathbb{C}}(V_1 \otimes C) = \dim_{\mathbb{R}} V_1\), we obtain \(\dim_{\mathbb{R}} V_1 = \sum_j \dim_{\mathbb{R}} U_j\). It follows that \(V_1 = U_1 \oplus \cdots \oplus U_n\). Lemma 4.2 implies \([U_j, U_{j'}] = 0\) for \(j \neq j'\). It remains to show that \(<U_j>\) is isomorphic to a complex Heisenberg algebra.

Recall that \(<W_j>\) is isomorphic to \(H_k^C\) for some \(k \geq 1\). Hence there are elements \(A_s, B_s \in W_j, 1 \leq s \leq k, Z \neq 0\) in the second layer of \(<W_j>\) such that

1. \([A_s, B_s] = Z\) for all \(s\), and \([A_s, A_s'] = [B_s, B_s'] = [A_s, B_{s'}] = 0\) for \(s \neq s'\).

By Lemma 4.2, \([W_j, W_{j'}]\) = 0 if \(j \neq j'\). Since \(\overline{W}_j = W_{j+n}\), we have \([A_s, \overline{A}_s] = [A_s, \overline{B}_s] = [\overline{A}_s, B_s] = [B_s, B_s] = 0\) for all \(s\), and \([A, B] = 0\) if \(s \neq s'\) and \(A \in \{A_s, B_s, \overline{A}_s, \overline{B}_s\}\) and \(B \in \{A_{s'}, B_{s'}, \overline{A}_{s'}, \overline{B}_{s'}\}\). Set

\[
X_s = \frac{1}{2}(A_s + \overline{A}_s), \quad \hat{X}_s = \frac{1}{2i}(A_s - \overline{A}_s);
\]

\[
Y_s = \frac{1}{2}(B_s + \overline{B}_s), \quad \hat{Y}_s = \frac{1}{2i}(B_s - \overline{B}_s);
\]

\[
Z_s = \frac{1}{4}(Z + \overline{Z}), \quad \hat{Z}_s = \frac{1}{4i}(Z - \overline{Z}).
\]

Since \(W_j\) and \(\overline{W}_j\) are distinct extended equivalence classes, we have \(CZ \cap CZ = \{0\}\). This implies that \(Z_s\) and \(\hat{Z}_s\) are linearly independent over \(\mathbb{R}\). Now it is easy to check the following:
(1) the following bracket relations hold:

\[
[X_s, Y_s] = Z_1, \quad [X_s, \tilde{Y}_s] = [\tilde{X}_s, Y_s] = \tilde{Z}_1, \quad [\tilde{X}_s, \tilde{Y}_s] = -Z_1, \quad [X_s, \tilde{X}_s] = [Y_s, \tilde{Y}_s] = 0;
\]

\[
[X, Y] = 0 \text{ if } s \neq s', \text{ and } X \in \{X_s, Y_s, \tilde{X}_s, \tilde{Y}_s\}, \; Y \in \{X_{s'}, Y_{s'}, \tilde{X}_{s'}, \tilde{Y}_{s'}\}.
\]

(2) \( <U_j> \) is 2-step, \( X_s, \tilde{X}_s, Y_s, \tilde{Y}_s, 1 \leq s \leq k, \) form a basis of \( U_j \), and \( Z_1, \tilde{Z}_1 \) form a basis of the second layer of \( <U_j> \).

It follows that \( <U_j> \) is isomorphic to the complex Heisenberg algebra \( \mathcal{H}_C \).

Further information about the Carnot algebras in Lemma 4.5 will be provided in Theorem 4.8. In fact, Theorem 4.8 gives a characterization of non-rigid Carnot algebras with irreducible first layer.

Heisenberg product algebras and complex Heisenberg product algebras are defined in the Introduction. We use the notation from there.

**Lemma 4.6.** Every Heisenberg product algebra \( \mathcal{N} \) has an irreducible first layer and satisfies \( r_1(\mathcal{N}) = 1 \). In particular, it is non-rigid.

**Proof.** Let \( \tilde{\mathcal{N}} \to \mathcal{N} \) be the natural projection. Then condition (1) in the definition of Heisenberg product algebra implies that the restriction \( P|_{\tilde{\mathcal{N}}} \) is a graded isomorphism from \( \tilde{\mathcal{H}}_j \) onto its image. Set \( L_j = P(\tilde{L}_j) \). Then \( \dim(L_j) = 1 \) and (2) implies \( L_i \cap L_j = \{0\} \) for \( i \neq j \).

We first show that \( r_1(\mathcal{N}) = 1 \). Let \( 0 \neq X \in V_1 \) be some \( \tilde{X} \in \tilde{V}_1 \). We claim that \( \text{rank}(X) \geq 1 \) always holds, and \( \text{rank}(X) = 1 \) if and only if \( \tilde{X} \in \tilde{V}_{1,j} \) for some \( j \). The claim implies \( r_1(\mathcal{N}) = 1 \) and so \( \mathcal{N} \) is non-rigid by Theorem 2.1. First assume \( \tilde{X} \in \tilde{V}_{1,j} \) for some \( j \). Since \( \text{rank}(\tilde{X}) = 1 \), we have \( \text{rank}(X) \leq 1 \). There is some \( Y \in \tilde{V}_{1,j} \) such that \( [\tilde{X}, Y] \neq 0 \). Since \( P|_{\tilde{\mathcal{N}}} \) is a graded isomorphism, we have \( [X, P(Y)] \neq 0 \), which implies \( \text{rank}(X) \geq 1 \). Hence \( \text{rank}(X) = 1 \).

Next we assume \( \tilde{X} \notin \cup_j \tilde{V}_{1,j} \). Write \( \tilde{X} = \tilde{X}_{j_1} + \cdots + \tilde{X}_{j_s} \) with \( s \geq 2 \) and \( 0 \neq \tilde{X}_{j_i} \in \tilde{V}_{1,j_i}, 1 \leq j_1 < \cdots < j_s \leq n \). There is some \( Y_{j_i} \in \tilde{V}_{1,j_i} \) satisfying \( 0 \neq [\tilde{X}_{j_i}, Y_{j_i}] \in L_{j_i} \). Then \( 0 \neq [\tilde{X}, Y_{j_i}] = [\tilde{X}_{j_i}, Y_{j_i}] \in \tilde{L}_{j_i} \). We have \( 0 \neq [X, P(Y_{j_i})] \in L_{j_i} \). Since \( L_i \cap L_j = \{0\} \) for \( i \neq j \), we see that \( [X, P(Y_{j_i})] \in L_{j_i} \) and \( [X, P(Y_{j_2})] \in L_{j_2} \) are linearly independent. Hence \( \text{rank}(X) \geq 2 \).

Next we show that \( \mathcal{N} \) has irreducible first layer. Notice that, each nonsingular linear transformation of the second layer of the Heisenberg algebra \( \mathcal{H}^n_R \), which is just multiplication by a nonzero constant, extends to a graded automorphism of \( \mathcal{H}^m_R \). Hence condition (4) in the definition of an Heisenberg product algebra implies each \( B \in G_2 \) extends to a graded automorphism of \( \tilde{\mathcal{N}} \). Now for each \( B \in G_2 \), fix some \( \tilde{B} \in \text{Aut}_g(\tilde{\mathcal{N}}) \) such that \( \tilde{B}|_{\tilde{V}_2} = B \).

Let \( \tilde{G} = \langle \tilde{B} : B \in G_2 \rangle \subset \text{Aut}_g(\tilde{\mathcal{N}}) \) be the group generated by all the \( \tilde{B}, B \in G_2 \). Notice that the \( \tilde{G} \)-action on \( \tilde{V}_2 \) factors through \( G_2 \). Therefore condition (3) implies that \( V \) is \( \tilde{G} \)-invariant. It follows that each \( \tilde{L} \in \tilde{G} \) induces a graded isomorphism \( L \in \text{Aut}_g(\mathcal{N}) \). Let \( G = \{L : \tilde{L} \in \tilde{G}\} \). Then \( G \) is a subgroup of \( \text{Aut}_g(\mathcal{N}) \) and the map \( \tilde{G} \to G, \tilde{L} \to L \) is a homomorphism. Condition (4) implies that \( G \) permutes the set \( \{P(\tilde{V}_{1,1}), \ldots, P(\tilde{V}_{1,n})\} \) and acts transitively on the set.
Let \( G_0 \subset \text{Aut}_g(\mathcal{H}_C^m) \) be the subgroup consisting of graded automorphisms that act trivially on the second layer. Notice that \( G_0 \) acts transitively on the set of nonzero vectors in the first layer (this follows from the proof of Lemma 4.4). Since \( G \subset \text{Aut}_g(\mathcal{N}) \) acts on the set \( \{ P(\tilde{V}_{1,j}), \ldots, P(\tilde{V}_{1,n}) \} \) transitively, we see that \( \text{Aut}_g(\mathcal{N}) \) acts transitively on the nonzero vectors in \( \cup_j P(\tilde{V}_{1,j}) \).

Now let \( U \subset V_1 \) be a non-trivial subspace of \( V_1 \) that is invariant under the action of \( \text{Aut}_g(\mathcal{N}) \). If \( U \cap P(\tilde{V}_{1,j}) \neq \{0\} \) for some \( j \), then \( P(\tilde{V}_{1,j}) \subset U \) for all \( j \) and so \( U = V_1 \). Now assume \( U \cap P(\tilde{V}_{1,j}) = \{0\} \) for all \( j \). Fix an inner product on \( V_1 \) and let \( S \subset V_1 \) be the unit sphere with respect to this inner product. Set \( K = S \cap (\cup_j P(\tilde{V}_{1,j})) \). Then \( \delta := d(U \cap S, K) > 0 \). Pick any \( 0 \neq x \in U \). Then \( x = P(\tilde{x}) \) for some \( \tilde{x} \in V_1 \). Write \( \tilde{x} = \tilde{x}_{j_1} + \cdots + \tilde{x}_{j_s} \) with \( s \geq 2 \) and \( 1 \leq j_1 < \cdots < j_s \leq n \) and \( 0 \neq \tilde{x}_{j_k} \in \tilde{V}_{1,j_k} \). There exists a sequence \( g_k \in G_0 \) such that \( g_k(\tilde{x}_{j_k}) = k\tilde{x}_{j_k} \). Define \( f_k \in \text{Aut}_g(\mathcal{N}) \) by \( f_k|_{\tilde{H}_{j_k}} = g_k \) and \( f_k|_{\tilde{R}_{j_k}} = id \) for \( j \neq j_k \). Notice that \( f_k \) pointwise fixes \( \tilde{V}_2 \). Hence it induces a graded automorphism \( F_k \) of \( \mathcal{N} \). We have \( F_k(x) = kP(\tilde{x}_{j_k}) + P(\tilde{x}_{j_2}) + \cdots + P(\tilde{x}_{j_s}) \). It is clear that \( d(\mathbb{R}F_k(x) \cap S, K) \to 0 \) as \( k \to \infty \). Since \( \mathbb{R}F_k(x) \subset U \), this contradicts \( \delta = d(U \cap S, K) > 0 \). Hence the first layer of \( \mathcal{N} \) is irreducible.

Notice that, the map \( \tau : \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \to \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \),

\[
\tau(w_1, \ldots, w_{2m}, z) = (\bar{w}_1, \ldots, \bar{w}_{2m}, \bar{z}),
\]

is a graded automorphism of \( \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \).

**Lemma 4.7.** Every complex Heisenberg product algebra \( \mathcal{N} \) has an irreducible first layer and satisfies \( r_1(\mathcal{N}) = 2 \), \( r_{1,C}(\mathcal{N}) = 1 \). In particular, it is non-rigid.

**Proof.** Let \( \tilde{\mathcal{N}} \to \mathcal{N} \) be the natural projection. Then condition (1) implies the restriction \( P|_{\tilde{H}_{j_k}} \) is a graded isomorphism from \( \tilde{H}_{j_k} \) onto its image. Set \( L_j = P(\tilde{L}_j) \). Then \( \text{dim}_\mathbb{R}(L_j) = 2 \). Recall that we have \( r_1(\mathcal{H}_C^m) = 2 \) and \( r_{1,C}(\mathcal{H}_C^m) = 1 \) for each complex Heisenberg algebra \( \mathcal{H}_C^m \). Since \( [P(\tilde{V}_{1,i}), P(\tilde{V}_{1,j})] = 0 \) for \( i \neq j \) and the restriction \( P|_{\tilde{H}_{j_k}} \) is a graded isomorphism from \( \tilde{H}_{j_k} \) onto its image, it is easy to see that \( r_1(\mathcal{N}) = 2 \), \( r_{1,C}(\mathcal{N}) = 1 \). By Theorem 2.1, \( \mathcal{N} \) is non-rigid.

Next we show that \( \mathcal{N} \) has irreducible first layer. First we notice that, if we identify \( \mathbb{C} \) with the second layer of \( \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \) and a map \( g : \mathbb{C} \to \mathbb{C} \) has the form \( g(z) = az \) or \( g(z) = az \bar{z} \) for some \( 0 \neq a \in \mathbb{C} \), then \( g \) extends to a graded isomorphism \( \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \to \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \). It follows that every \( g_2 \in G_2 \) extends to a graded automorphism of \( \mathcal{N} \). Then the arguments in the last 3 paragraphs of the proof of Lemma 4.6 show that \( \mathcal{N} \) has an irreducible first layer.

Finally we are ready to characterize the non-rigid Carnot algebras with irreducible first layer.
Then and second layers of \( f \) further,

\\[ U(1) \]

exactly one of the following happens:

1. \( N \) is abelian;
2. \( r_1(N) = 1 \) and \( N \) is a Heisenberg product algebra;
3. \( r_1(N) = 2, r_{1,c}(N) = 1 \) and \( N \) is a complex Heisenberg product algebra.

**Proof.** An abelian Carnot algebra clearly is non-rigid and has irreducible first layer. Lemmas 4.6 and 4.7 imply Heisenberg product algebras and complex Heisenberg product algebras are non-rigid and have irreducible first layer.

Conversely, let \( N \) be a non-abelian non-rigid Carnot algebra with irreducible first layer. We need to show that \( N \) satisfies either (2) or (3). Lemma 3.1 and Theorem 2.1 imply that we have one of the following two cases: (1) \( r_1(N) = 1 \); (2) \( r_1(N) \geq 2 \) and \( r_{1,c}(N) \leq 1 \).

**Case (1) \( r_1(N) = 1 \).**

By Lemma 4.3 and Lemma 4.2, \( V_1 \) can be written as a direct sum of vector subspaces \( V_1 = U_1 \oplus \cdots \oplus U_n \), where each \( U_j \) is an extended equivalence class, \([U_i, U_j] = 0\) for \( i \neq j\); furthermore, \( U_j \) is maximal among linear subspaces consisting of 0 and rank 1 elements, and \( \langle U_j \rangle \) is isomorphic to an Heisenberg algebra.

Since \( U_j \) is maximal among linear subspaces consisting of 0 and rank 1 elements, the action of \( Aut(N) \) on \( V_1 \) permutes the subalgebras \( U_1, \ldots, U_n \). Let \( U \) be the direct sum of all those \( U_j \) in the orbit of \( U_1 \). Then \( U \) is invariant under the action of \( Aut(N) \). Since \( V_1 \) is irreducible, we must have \( U = V_1 \). Hence \( Aut(N) \) acts transitively.

It follows that all the subalgebras \( \langle U_j \rangle \) are isomorphic. Suppose they are isomorphic to \( H^m_\mathbb{R} \) for some \( m \geq 1 \).

Let \( \bar{N} = \bar{H}_1 \oplus \cdots \oplus \bar{H}_n \), where each \( \bar{H}_j = \bar{H}^m_\mathbb{R} \). Let \( \bar{V}_1 \) and \( \bar{V}_2 \) respectively be the first and second layers of \( \bar{N} \). Let \( \bar{V}_1 \) and \( \bar{V}_2 \) respectively be the first and second layers of \( \bar{N} \). Then \( \bar{V}_1 = \bar{V}_{1,1} \oplus \cdots \oplus \bar{V}_{1,n} \) and \( \bar{V}_2 = \bar{L}_1 \oplus \cdots \oplus \bar{L}_n \). For each \( j \), let \( f_j : \bar{H}_j \to \langle U_j \rangle \) be a fixed graded isomorphism. We also use \( f_j \) to denote the composition of \( f_j : \bar{H}_j \to \langle U_j \rangle \) with the inclusion \( \langle U_j \rangle \subset N \). Now define \( P : \bar{N} \to N \) by \( P(x_1, \ldots, x_n) = \sum_j f_j(x_j) \).

Using the facts that \([U_i, U_j] = 0\) for \( i \neq j\) and that \( V_1 = U_1 \oplus \cdots \oplus U_n \), it is easy to check that \( P \) is a surjective graded homomorphism, and \( V := \ker P \subset \bar{V}_2 \). It follows that \( N \cong \bar{N}/V \).

As \( P|_{\bar{H}_j} = f_j \) is an isomorphism, we have \( V \cap \bar{L}_j = \{0\} \). Set \( L_j = f_j(\bar{L}_j) \). Note that \( L_j \) is the second layer of \( \langle U_j \rangle \). Since \( U_i \) and \( U_j \) are distinct extended equivalence classes for \( i \neq j \), we have \( L_i \cap L_j = \{0\} \). This implies \( (V + \bar{L}_i) \cap (V + \bar{L}_j) = V \) for \( i \neq j \).

Since \( Aut(N) \) permutes the subalgebras \( \langle U_j \rangle \) and \( P|_{\bar{H}_j} = f_j \) is an isomorphism from \( \bar{H}_j \) onto \( \langle U_j \rangle \), for each \( A \in Aut(N) \), there is a unique lift \( \bar{A} : \bar{H}_j \to \bar{N} \) for \( A|_{\langle U_j \rangle} : \langle U_j \rangle \to N \). In fact, if \( A(\langle U_j \rangle) = \langle U_{\sigma(j)} \rangle \) for some \( \sigma(j) \), then \( \bar{A} = f_{\sigma(j)}^{-1} \circ A \circ f_j \). Define \( \bar{A} : \bar{N} \to \bar{N} \) by \( \bar{A}(x_1, \ldots, x_n) = \sum_j \bar{A}(x_j) \). Then it is easy to see that \( \bar{A} \) is a graded isomorphism and is a lift of \( A \) (i.e., \( P \circ \bar{A} = \bar{A} \circ P \)). Furthermore, it is also easy to check that \( \bar{A} \) is the only graded isomorphism that lifts \( A \). It follows that the map \( h : Aut(N) \to Aut(\bar{N}) \), \( h(A) = \bar{A} \), is an injective homomorphism. Set \( G = h(Aut(N)) \).

Then \( V \) is \( G \)-invariant. Since \( Aut(N) \) acts transitively on the set \( \{U_1, \ldots, U_n\} \), \( G \) acts transitively on the set \( \{\bar{H}_1, \ldots, \bar{H}_n\} \).

Let \( R : Aut(N) \to GL(\bar{V}_2) \) be defined by \( R(\bar{A}) = \bar{A}|_{\bar{V}_2} \). Set \( G_2 = R(G) \). Clearly the \( G \)
action on \( \tilde{V}_2 \) factors through \( G_2 \). Hence \( V \subset \tilde{V}_2 \) is \( G_2 \)-invariant and \( G_2 \) acts transitively on the set \( \{ \tilde{L}_1, \cdots, \tilde{L}_n \} \) of “coordinate axes” of \( \tilde{V}_2 \).

**Case (2)** \( r_1(\mathcal{N}) \geq 2 \) and \( r_{1, \mathcal{C}}(\mathcal{N}) \leq 1 \).

The proof is similar to that of case (1). We will mainly indicate the difference.

Lemma 4.5 implies \( V_1 \) can be written as a direct sum of vector subspaces \( V_1 = U_1 \oplus \cdots \oplus U_n \) such that [\( U_j, U_{j'} \)] = 0 for \( j \neq j' \), and for each \( j, < U_j > \) is isomorphic to a complex Heisenberg algebra.

Recall that (see the proof of Lemma 4.5) (a) \( U_j = \pi_1(W_j) \), where \( W_j \) is an extended equivalence class of \( V_1 \otimes \mathbb{C} \); (b) each extended equivalence class is a maximal linear subspace (over \( \mathbb{C} \)) of \( V_1 \otimes \mathbb{C} \) consisting of (0 and) rank 1 elements. Hence \( \text{Aut}_g(\mathcal{N}) \) permutes the \( W_j \)'s. Notice that for any \( A \in \text{Aut}_g(\mathcal{N}) \), we have \( A \circ \pi_1 = \pi_1 \circ A \). Suppose \( A(W_j) = W_{\sigma(j)} \) for some \( \sigma(j) \). Then \( A(U_j) = U_{\sigma(j)} \). So \( \text{Aut}_g(\mathcal{N}) \) permutes the \( U_j \)'s. Since the action of \( \text{Aut}_g(\mathcal{N}) \) on \( V_1 \) is irreducible, we see that \( \text{Aut}_g(\mathcal{N}) \) acts transitively on the set \( \{ U_1, \cdots, U_n \} \).

It follows that all the subalgebras \( < U_j > \) are isomorphic. Suppose they are isomorphic to \( \mathcal{H}_C^m \) for some \( m \geq 1 \).

Let \( \mathcal{N} = \tilde{H}_1 \oplus \cdots \oplus \tilde{H}_n \), where each \( \tilde{H}_j = \mathcal{H}_C^m \). Let \( \tilde{V}_{1,j} \) and \( \tilde{L}_j \) respectively be the first and second layers of \( \tilde{H}_j \). Let \( \tilde{V}_1 \) and \( \tilde{V}_2 \) respectively be the first and second layers of \( \mathcal{N} \). Then \( \tilde{V}_1 = \tilde{V}_{1,1} \oplus \cdots \oplus \tilde{V}_{1,n} \) and \( \tilde{V}_2 = \tilde{L}_1 \oplus \cdots \oplus \tilde{L}_n \). For each \( j \), let \( f_j : \tilde{H}_j \rightarrow < U_j > \) be a fixed graded isomorphism. As in Case (1) we define a surjective graded homomorphism \( P : \mathcal{N} \rightarrow \mathcal{N} \). Let \( V = \ker P \). Then \( V \subset \tilde{V}_2 \) and \( \mathcal{N} \cong \mathcal{N}/V \). Since \( P|_{\tilde{H}_j} = f_j \) is an isomorphism, we see that \( \tilde{L}_j \cap V = \{ 0 \} \).

As in the proof of Case (1), we construct groups \( G \subset \text{Aut}_g(\mathcal{N}) \) and \( G_2 \subset GL(\tilde{V}_2) \). The arguments there show that \( V \) is \( G_2 \)-invariant, \( G_2 \) permutes the set \( \{ L_1, \cdots, L_n \} \) and acts transitively on the set. Recall that each \( g_2 \in G_2 \) is the restriction \( g|_{\tilde{V}_2} \) of some \( g \in G \subset \text{Aut}_g(\mathcal{N}) \). Notice that \( g \) permutes the \( \tilde{H}_j \)'s and for each \( j \), \( g|_{\tilde{H}_j} : \tilde{H}_j \rightarrow \tilde{H}_{\sigma(j)} \) is an isomorphism of the complex Heisenberg algebra. By [S], for every graded automorphism \( A \) of the complex Heisenberg algebra \( \mathcal{H}_C^m = \mathbb{C}^{2m} \oplus \mathbb{C} \), the map \( B := A|_{\{ 0 \} \oplus \mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \) has the form \( B(z) = az \) or \( B(z) = az^2 \) for some \( 0 \neq a \in \mathbb{C} \). Hence \( V \) and \( G_2 \) also satisfy condition (4) in the definition of the complex Heisenberg product algebra.

\[ \square \]

### 5 Rigidity of quasisymmetric maps

The goal of this Section is to show that non-rigid Carnot algebras with reducible first layer are quasisymmetrically rigid. We consider two cases depending on whether the invariant subspace is abelian.

#### 5.1 When the invariant subspace is non-abelian

In this Subsection we consider the case when the invariant subspace is non-abelian.

**Lemma 5.1.** Let \( \mathcal{N} \) be a \( r \)-step Carnot algebra with \( r \geq 3 \). If \( r_1(\mathcal{N}) = 1 \) and \( W := < W_1 > \) is not abelian, then \( \mathcal{N} \) is quasisymmetrically rigid.
Proof. By Corollary 3.4, \( W_1 \) is a non-trivial proper subspace of \( V_1 \) invariant under the action of \( \text{Aut}_g(\mathcal{N}) \). Since \( r_1(\mathcal{N}) = 1 \) and \( W_1 \) is spanned by rank 1 elements, we have \( r_1(\mathcal{W}) \leq 1 \). First assume \( r_1(\mathcal{W}) = 0 \). Lemma 3.2 implies \( \mathcal{W} = \mathbb{R}^m \oplus \mathcal{W} \) is a direct sum of an Euclidean algebra and another Carnot algebra. Since \( \mathcal{W} \) is not abelian, \( \mathcal{W} \) is not Euclidean. Then the main result in [X3] implies \( \mathcal{W} \) is quasisymmetrically rigid. Now Theorem 2.4 implies \( \mathcal{N} \) is quasisymmetrically rigid.

Next we assume \( r_1(\mathcal{W}) = 1 \). We claim that \([X, W_1] \subset \mathcal{W}\) for any \( X \in V_1 \). The Lemma then follows from Theorem 2.3. Indeed, by definition \( \mathcal{W} \) has a vector space basis \( \{v_1, \ldots, v_m\} \) such that each \( v_i \) has rank 1 in the algebra \( \mathcal{N} \). Since \( r_1(\mathcal{W}) = 1 \), \( v_i \) also has rank 1 in the algebra \( \mathcal{W} \). Hence there is some \( w_i \in W_1 \) such that \([v_i, w_i] \neq 0\). Since \( v_i \) has rank 1 in \( \mathcal{N} \), for any \( X \in V_1 \), we have \([v_i, X] \in \mathbb{R}[v_i, w_i] \subset [W_1, W_1] \). Since this holds for all \( i \), we see that \([X, W_1] \subset [W_1, W_1]\) for all \( X \in V_1 \). Hence the claim holds.

The subspace \( \hat{W}_1 \) of \( V_1 \) is defined in Corollary 3.7.

**Lemma 5.2.** Let \( \mathcal{N} \) be a \( r \)-step Carnot algebra with \( r \geq 3 \). If \( r_1(\mathcal{N}) \geq 2 \), \( r_1, \mathcal{C}(\mathcal{N}) \leq 1 \), and \( \hat{W} := \langle \hat{W}_1 \rangle \) is not abelian, then \( \mathcal{N} \) is quasisymmetrically rigid.

**Proof.** Lemma 3.5 and Lemma 3.6 imply that \( r_1(\mathcal{N}) = 2 \) and \( r_1, \mathcal{C}(\mathcal{N}) = 1 \). Corollary 3.7 implies \( \hat{W}_1 \) is a non-trivial proper subspace of \( V_1 \) invariant under the action of \( \text{Aut}_g(\mathcal{N}) \). In the proof of Lemma 3.6, we showed that if \( X = X_1 + iX_2 \in V_1 \odot \mathbb{C} \) satisfies \( \text{rank}(X) = 1 \), then \( \text{rank}(X_1) \leq 2 \). Since \( r_1(\mathcal{N}) = 2 \), we have either \( X_1 = 0 \) or \( \text{rank}(X_1) = 2 \). Since \( W_1, \mathcal{C} \) is spanned by rank one elements in \( V_1 \odot \mathbb{C} \), we see that \( \hat{W}_1 \) is spanned by rank two elements in \( \mathcal{N} \). It follows that \( r_1(\hat{W}) \leq 2 \).

**Claim** \( r_1(\hat{W}) = 0 \) or \( r_1(\hat{W}) = 2 \).

We shall first assume the claim and finish the proof of the Lemma, then prove the claim at the end. First assume \( r_1(\hat{W}) = 0 \). Since \( \hat{W} \) is not abelian, the main result in [X3] implies \( \hat{W} \) is quasisymmetrically rigid. Now Theorem 2.4 implies \( \mathcal{N} \) is quasisymmetrically rigid.

Next we assume \( r_1(\hat{W}) = 2 \). We claim that \([X, \hat{W}_1] \subset \hat{W}\) for any \( X \in V_1 \). The Lemma then follows from Theorem 2.3. Indeed, as observed above, \( \hat{W}_1 \) has a vector space basis \( \{v_1, \ldots, v_m\} \) such that each \( v_i \) has rank 2 in the algebra \( \mathcal{N} \). Since \( r_1(\hat{W}) = 2 \), \( v_i \) also has rank 2 in the algebra \( \hat{W} \). It follows that \([v_i, \mathcal{N}] = [v_i, \hat{W}] \subset \hat{W} \). Since this holds for all \( i \), we see that \([\hat{W}_1, \mathcal{N}] \subset \hat{W} \). Hence the claim holds.

Now we prove the claim. We assume \( r_1(\hat{W}) \geq 1 \) and need to show \( r_1(\hat{W}) = 2 \). Note that if \( A \) is an extended equivalence class in \( V_1 \odot \mathbb{C} \), then so is its complex conjugate \( \bar{A} \). Furthermore, \( A \oplus \bar{A} = U + iU \), where

\[
U = \pi_1(A) = \{X \in V_1 : X + iY \in A \text{ for some } Y \in V_1\}.
\]

Let \( 0 \neq X_1 \in U \). Then \( X_1 + iX_2 \in A \) for some \( X_2 \in V_1 \). In the proof of Lemma 3.6 we showed that \( \text{rank}(X_1) \leq 2 \). Our assumption \( r_1(\mathcal{N}) \geq 2 \) now implies \( \text{rank}(X_1) = 2 \). Since \( \text{rank}(X_1 + iX_2) = 1 \), we have \([X_1 + iX_2, V_1] \neq 0\), otherwise \([X_1 + iX_2, \mathcal{N} \odot \mathbb{C}] = 0 \). Let \( Y_0 \in V_1 \) be an arbitrary element such that \([X_1 + iX_2, Y_0] \neq 0\). Then \([X_1 + iX_2, \mathcal{N}] \subset [X_1 + iX_2, \mathcal{N} \odot \mathbb{C}] = \mathbb{C}[X_1 + iX_2, Y_0] \), which implies \([X_1, \mathcal{N}] \subset \mathbb{R}[X_1, Y_0] \oplus \mathbb{R}[X_2, Y_0] \). Since \( \text{rank}(X_1) = 2 \), we see that \([X_1, Y_0]\) and \([X_2, Y_0]\) are linearly independent (over \( \mathbb{R} \)).
Notice that $W_{1,\mathbb{C}}$ is spanned by extended equivalence classes in $V_1 \otimes \mathbb{C}$. There are extended equivalence classes $A_1, \ldots, A_n$ such that $W_{1,\mathbb{C}} = (A_1 + \tilde{A}_1) + \cdots + (A_n + \tilde{A}_n)$ and $A_i \neq A_j, \tilde{A}_j$ for $i \neq j$. Let $U_j = \pi_1(A_j)$. Then $A_j \oplus \tilde{A}_j = U_j \oplus iU_j$ and $\tilde{W}_1 = U_1 + \cdots + U_n$. Lemma 4.2 implies $[U_i, U_j] = 0$ for $i \neq j$. Since $r_1(\tilde{W}) \geq 1$, for any $0 \neq Y_0 \in U_j$, there must exist some $X_1 \in U_j$ such that $[X_1, Y_0] \neq 0$. The preceding paragraph now implies $[Y_0, U_j] = [Y_0, N]$ has dimension 2. Now any $0 \neq Y \in \tilde{W}_1$ can be written as $Y = Y_{j_1} + \cdots + Y_{j_s}$, where $1 \leq j_1 < \cdots < j_s \leq n$, and $0 \neq Y_{j_i} \in U_{j_i}$. It follows that $[Y, \tilde{W}_1] \supset [Y, U_{j_1}] = [Y_{j_1}, U_{j_1}]$ has dimension at least 2. Hence $Y$ has rank at least 2 in $\tilde{W}$.

\[ \square \]

### 5.2 When invariant subspace is abelian

In this subsection we first treat the case when the invariant subspace is abelian, then finish the proof of Theorem 1.1.

The following lemma follows from the BCH formula (see the proof of Lemma 2.1 in [X1]).

**Lemma 5.3.** There are universal constants $c_j \in \mathbb{Q}$ for $j \geq 1$ with the following property: $c_1 = 1/2$, and if $N$ is $r$-step and $Y \in N$ satisfies $[Y, V_i] = 0$ for all $i \geq 2$, then for any $X \in N$ we have:

\[ X * Y = X + Y + \sum_{j=1}^{r-1} c_j(ad X)^j Y \]

(5.1)

and

\[ Y * X = X + Y + \sum_{j=1}^{r-1} (-1)^j c_j(ad X)^j Y. \]

(5.2)

**Lemma 5.4.** Let $W \subset V_1$ be a linear subspace satisfying $[W, W] = 0$ and $[W, V_i] = 0$ for all $i \geq 2$. Let $\tilde{W} \subset V_1$ be a subspace complementary to $W$. Then for any $u \in W$ and any $x = x_1 + \sum_{j=1}^{r} \tilde{x}_j \ (x_1 \in W, \tilde{x}_1 \in \tilde{W}, \tilde{x}_j \in V_j \text{ for } j \geq 2)$, we have $(ad x)^i u = (ad \tilde{x}_1)^i u$ for all $i \geq 1$.

**Proof.** We first show that for any integer $k \geq 0$, any $j \geq 2$ and any $y_j \in V_j$ we have

\[ [y_j, (ad \tilde{x}_1)^k u] = 0. \]

(5.3)

We induct on $k$. The base case $k = 0$ follows from the assumption that $[W, V_j] = 0$ for $j \geq 2$. Assume the claim holds for $k$. Now the Jacobi identity implies

\[ [y_j, (ad \tilde{x}_1)^{k+1} u] = [y_j, (ad \tilde{x}_1)^k u] + \sum_{j=1}^{r} [y_j, (ad \tilde{x}_1)^k u]. \]

Notice that $[y_j, \tilde{x}_1] \in V_{j+1}$. The induction hypothesis now implies $[[y_j, \tilde{x}_1], (ad \tilde{x}_1)^k u] = 0$ and $[y_j, (ad \tilde{x}_1)^k u] = 0$.

Next we prove $(ad \tilde{x})^i u = (ad \tilde{x}_1)^i u$ by inducting on $i$. The base case $i = 1$ follows from the assumptions $[W, W] = 0$ and $[W, V_j] = 0$ for $j \geq 2$:

\[ (ad \tilde{x}) u = [\tilde{x}, u] = [x_1, u] + [\tilde{x}_1, u] + \sum_{j=2}^{r} [\tilde{x}_j, u] = [\tilde{x}_1, u] + 0 = (ad \tilde{x}_1) u. \]
Now assume \((ad \tilde{x})^i u = (ad \tilde{x}_1)^i u\) holds for \(i\). Then the induction hypothesis, the condition \([W, V_j] = 0\) for \(j \geq 2\) and (5.3) imply
\[
(ad \tilde{x})^{i+1} u = [\tilde{x}, (ad \tilde{x})^i u] = [\tilde{x}, (ad \tilde{x}_1)^i u]
\]
\[
= [x_1, (ad \tilde{x}_1)^i u] + [\tilde{x}_1, (ad \tilde{x}_1)^i u] + \sum_{j=2}^r [\tilde{x}_j, (ad \tilde{x}_1)^i u]
\]
\[
= 0 + (ad \tilde{x}_1)^{i+1} u + 0 = (ad \tilde{x}_1)^{i+1} u.
\]

\[\square\]

**Theorem 5.5.** Let \(\mathcal{N} = V_1 \oplus \cdots \oplus V_r\) be a Carnot algebra, \(W \subset V_1\) a non-trivial proper subspace invariant under the action of \(\text{Aut}_q(\mathcal{N})\). If \([W, W] = 0\) and \([W, V_i] = 0\) for all \(i \geq 2\), then \(\mathcal{N}\) is quasisymmetrically rigid.

Let \(\tilde{W} \subset V_1\) be a subspace complementary to \(W\). We fix an inner product \(|\cdot|\) on each of \(W, \tilde{W}, V_j, j \geq 2\). Then there exists some constant \(A \geq 1\) such that
\[
||v, w|| \leq A \cdot |v| \cdot |w|
\]
for all \(v, w \in W \cup \tilde{W} \cup \bigcup_{j=2}^r V_j\).

Define a norm on \(\mathcal{N}\) by
\[
||x|| = |x_1| + \sum_{j=1}^r |\tilde{x}_j|^\frac{1}{2}
\]
for \(x = x_1 + \tilde{x}_1 + \cdots + \tilde{x}_r \in \mathcal{N}\) with \(x_1 \in W, \tilde{x}_1 \in \tilde{W}, \tilde{x}_j \in V_j\) for \(j \geq 2\).

We recall that, if \(F : X \to Y\) is \(\eta\)-quasisymmetric, then \(F^{-1} : Y \to X\) is \(\eta_1\)-quasisymmetric with \(\eta_1(t) = (\eta^{-1}(t^{-1}))^{-1}\). Without loss of generality we may assume \(\eta(1) \geq 1\). It follows that \(\eta_1(1) \geq 1\). These inequalities will be used implicitly.

The condition \([W, W] = 0\) implies \(W\) is a Lie subalgebra of \(\mathcal{N}\). We shall abuse notation and also denote by \(W\) the connected Lie subgroup of \(\mathcal{N}\) with Lie algebra \(W \subset \mathcal{N}\). Here \(\mathcal{N}\) denotes the Carnot group with Lie algebra \(\mathcal{N}\).

Let \(d\) be the homogeneous distance on \(\mathcal{N}\) associated with the above defined norm on \(\mathcal{N}\). That is, \(d(x, y) = ||(-x) * y||\) for \(x, y \in \mathcal{N} = \mathcal{N}\). Let \(F : (N, d) \to (N, d)\) be an \(\eta\)-quasisymmetric map for some \(\eta\). Since \(W \subset \mathcal{N}\) is an invariant subspace, Proposition 2.5 implies that \(F\) sends left cosets of \(W\) to left cosets of \(W\).

Recall that \(l_F\) and \(L_F\) are defined in Section 2.3.

**Lemma 5.6.** Let \(L\) be a left coset of \(W\). Suppose \(p, q \in L\) are such that \(l_F(p) > C_1 \cdot L_F(q)\) with \(C_1 = 200(r + 1)A^{-1}C\eta_1(1)\), where \(C = 1 + \max_{1 \leq j \leq r-1} |c_j|\) and the \(c_j\)'s are the constants in Lemma 5.3. Write \(q = p * u\) with \(u \in W\). Then for any \(s = p * tu\) with \(|t| \geq 1\), we have
\[
L_F(s) \leq \frac{2(r + 1)CA^{-1}(\eta_1(1))^2}{|t|^\frac{1}{2}} \cdot L_F(q).
\]

19
Proof. The proof is a modification of the proof of Lemma 5.1, [X2].

Fix some \( e \in V_2 \) with \(|e| = 1\). Set \( p' = F(p), q' = F(q) \) and \( L' = F(L) = p' \ast W \). Let \( \{r_j\} \) be an arbitrary sequence of positive reals such that \( r_j \to 0 \). Denote \( \bar{p}_j = p' \ast r_j^2 e \) and \( \bar{q}_j = q' \ast r_j^2 e \). Note \( p', q' \in L' \). So \( q' = p' \ast u' \) for some \( u' \in W \). Since \([W, V_2] = 0\), we have \( u' \ast r_j^2 e = r_j^2 e \ast u' \). It follows that

\[
\bar{q}_j = p' \ast u' \ast r_j^2 e = p' \ast r_j^2 e \ast u' = \bar{p}_j \ast u' \in \bar{p}_j \ast W.
\]

Hence \( \bar{p}_j \) and \( \bar{q}_j \) lie on the same left coset of \( W \). Let \( p_j \) and \( q_j \) be points on \( F^{-1}(\bar{p}_j \ast W) \) nearest to \( p \) and \( q \) respectively.

Since \( l_F(p) > C_1 \cdot L_F(q) \), the first part of the proof of Lemma 5.1, [X2] yields

\[
d(p, p_j) \leq \frac{1}{101(r + 1)CA^{r-1}}d(q, q_j) \tag{5.5}
\]

for all sufficiently large \( j \).

Next we shall look at \( d(p, p_j) \) and \( d(q, q_j) \).

Since \( q, p \) lie on the same left coset, we can write \( q = p \ast u \) for some \( u \in W \). Similarly, \( q_j = p_j \ast w \) for some \( w \in W \). Let \( o_j = p_j \ast w' \ (w' \in W) \) be an arbitrary point on the left coset \( p_j \ast W \). Also write \( p_j = p \ast (x_1 + \tilde{x}), \) where \( x_1 \in W \) and \( \tilde{x} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_r \) with \( \tilde{x}_1 \in W \) and \( \tilde{x}_i \in V_i \) for \( i \geq 2 \). Although the \( x_1 \) and \( \tilde{x}_i \)'s depend on \( r_j \), we shall suppress the dependence to simplify the notation.

Now we calculate \( d(p, p_j) \) and \( d(q, o_j) \). Notice

\[
(-p) \ast p_j = (-p) \ast p \ast (x_1 + \tilde{x}) = x_1 + \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_r.
\]

So

\[
d(p, p_j) = ||(-p) \ast p_j|| = |x_1| + \sum_{j=1}^{r} |\tilde{x}_j|^{\frac{1}{2}}. \tag{5.6}
\]

Using Lemma 5.3, the conditions \([W, W] = 0\), \([W, V_i] = 0\) for \( i \geq 2 \), and Lemma 5.4, we
obtain:

\[
(-q) \ast o_j = (-q) \ast p_j \ast w' = (-u) \ast (-p) \ast p \ast (x_1 + \tilde{x}) \ast w' = (-u) \ast (x_1 + \tilde{x}) \ast w' \\
= \left( (x_1 - u) + \tilde{x} - \sum_{j=1}^{r-1} (-1)^j c_j (ad (x_1 + \tilde{x}))^j u \right) \ast w' \\
= \left( (x_1 - u) + \tilde{x} - \sum_{j=1}^{r-1} (-1)^j c_j (ad \tilde{x}_1)^j u \right) \ast w' \\
= (x_1 - u + w') + \tilde{x} - \sum_{j=1}^{r-1} (-1)^j c_j (ad \tilde{x}_1)^j u + \sum_{j=1}^{r-1} c_j (ad \tilde{x}_1)^j w' \\
= (x_1 - u + w') + \tilde{x}_1 + \sum_{j=1}^{r-1} (\tilde{x}_{j+1} + c_j (ad \tilde{x}_1)^j (w' - (-1)^j u)) \right].
\]

Hence

\[
d(q, o_j) = |x_1 - u + w'| + |\tilde{x}_1| + \sum_{j=1}^{r-1} |\tilde{x}_{j+1} + c_j (ad \tilde{x}_1)^j (w' - (-1)^j u)|^{\frac{1}{r+1}} .
\]

(5.7)

Since \( j \) is a point on \( p_j \ast W \) nearest to \( q \), we have \( d(q_j, q) \leq d(o_j, q) \) for any \( o_j \in p_j \ast W \). By (5.5), we have \( d(p, p_j) \leq d(q, o_j)/(101(r+1)CA^{r-1}) \). In particular, this inequality holds for the point \( o_j = p_j \ast u \). Set \( a = d(q, p_j \ast u) \). Notice that

\[
a = d(q, p_j \ast u) = |x_1| + |\tilde{x}_1| + \sum_{j: \text{even}} |\tilde{x}_{j+1}|^{\frac{1}{r+1}} + \sum_{j: \text{odd}} |\tilde{x}_{j+1} + 2c_j (ad \tilde{x}_1)^j u|^{\frac{1}{r+1}},
\]

(5.8)

where \( j \) in the above sums varies from 1 to \( r - 1 \). Since \( d(p, p_j) \leq a/(101(r+1)CA^{r-1}) \), from (5.6) we get

\[
|x_1| \leq a/(101(r+1)CA^{r-1}) \quad \text{and} \quad |\tilde{x}_j|^{\frac{1}{2}} \leq a/(101(r+1)CA^{r-1})
\]

(5.9)

for all \( 1 \leq j \leq r \). Now (5.8) and (5.9) imply

\[
|\tilde{x}_{j_0+1} + 2c_{j_0} (ad \tilde{x}_1)^{j_0} u|^{\frac{1}{r+1}} \geq a/(r+1)
\]

(5.10)

for some odd \( j_0, 1 \leq j_0 \leq r - 1 \). The triangle inequality and (5.9), (5.10) imply

\[
|2c_{j_0} (ad \tilde{x}_1)^{j_0} u| \geq \left( 1 - \frac{1}{(101CA^{r-1})^{j_0+1}} \right) \left( \frac{a}{r+1} \right)^{j_0+1}.
\]

(5.11)

Next we consider a point \( s \in p \ast W \) of the form \( s = p \ast tu \), where \( |t| \geq 1 \). We claim that

\[
d(s, p_j \ast W) \geq \frac{|t|^{\frac{1}{j_0+1}}}{2(r+1)CA^{r-1}} \cdot d(q, q_j).
\]

(5.12)
In the formula (5.7) we replace $u$ with $tu$ and $q$ with $s$ to obtain:

$$d(s, o_j) = |x_1 - tu + w'| + |\tilde{x}_1| + \sum_{j=1}^{r-1} |\tilde{x}_{j+1} + c_j(ad \tilde{x}_1)^j(w' - (-1)^jtu)|^{1/\tilde{r}}. \quad (5.13)$$

Set $v = w' - tu$. We divide points on $p_j \ast W$ into two types, depending on whether $|v| \geq |t|^{1/\tilde{r} + 1} \cdot \frac{a}{(r+1)^{CA^{r-1}}}$. If $|v| \geq |t|^{1/\tilde{r} + 1} \cdot \frac{a}{(r+1)^{CA^{r-1}}}$, then (5.13) and (5.9) imply

$$d(s, o_j) \geq |x_1 - tu + w'| = |x_1 + v| \geq |t|^{1/\tilde{r} + 1} \cdot \frac{a}{2(r + 1)CA^{r-1}} \geq |t|^{1/\tilde{r} + 1} \cdot \frac{d(q, q_j)}{2(r + 1)CA^{r-1}}. \quad (5.14)$$

Now assume $|v| \leq |t|^{1/\tilde{r} + 1} \cdot \frac{a}{(r+1)^{CA^{r-1}}}$. In this case,

$$d(s, o_j) \geq |\tilde{x}_{j0+1} + c_{j0}(ad \tilde{x}_1)^{j0}(w' - (-1)^{j0}tu)|^{1/\tilde{r}}. \quad (5.15)$$

Notice

$$w' - (-1)^{j0}tu = w' + tu = 2tu + v,$$

so

$$\tilde{x}_{j0+1} + c_{j0}(ad \tilde{x}_1)^{j0}(w' - (-1)^{j0}tu) = \tilde{x}_{j0+1} + 2tc_{j0}(ad \tilde{x}_1)^{j0}u + c_{j0}(ad \tilde{x}_1)^{j0}v.$$

Now (5.4) and (5.9) imply

$$|c_{j0}(ad \tilde{x}_1)^{j0}v| \leq |c_{j0}| \cdot A^{j0} \cdot |x_1|^{j0} \cdot |v|$$

$$\leq C \cdot A^{j0} \cdot \left(\frac{a}{101(r+1)CA^{r-1}}\right)^{j0} \cdot |t|^{1/\tilde{r} + 1} \cdot \frac{a}{(r+1)CA^{r-1}}$$

$$\leq \frac{1}{101} \cdot \left(\frac{a}{r+1}\right)^{j0+1} \cdot |t|^{1/\tilde{r} + 1}. \quad (5.16)$$

Since $|t| \geq 1$, (5.9), (5.11) and (5.16) imply

$$|\tilde{x}_{j0+1} + c_{j0}(ad \tilde{x}_1)^{j0}(w' - (-1)^{j0}tu)|$$

$$\geq \left(\frac{a}{r+1}\right)^{j0+1} \cdot \left(|t| \cdot (1 - \frac{1}{101CA^{r-1})^{j0+1}} - \frac{1}{101} \cdot \frac{1}{(101CA^{r-1})^{j0+1}}\right)$$

$$\geq \frac{|t|}{2} \cdot \left(\frac{a}{r+1}\right)^{j0+1}. \quad (5.17)$$

Hence by (5.15) we have

$$d(s, o_j) \geq \frac{a}{(r+1)^2} \cdot |t|^{1/\tilde{r} + 1} \geq \frac{d(q, q_j)}{(r+1)^2} \cdot |t|^{1/\tilde{r} + 1}. \quad (5.17)$$

Now (5.12) follows from (5.14) and (5.17). The argument at the end of the proof of Lemma 5.1 in [X2] now implies

$$L_F(s) \leq \frac{2(r+1)CA^{r-1}(\eta_1(1))^2}{|t|^{1/\tilde{r} + 1}} \cdot L_F(q) \leq \frac{2(r+1)CA^{r-1}(\eta_1(1))^2}{|t|^{1/\tilde{r} + 1}} \cdot L_F(q).$$
The rest of the proof of Theorem 5.5 is the same as in Section 5 of [X2].

The subspaces $W_1$ and $\hat{W}_1$ of $V_1$ are defined in Section 3.

**Lemma 5.7.** Let $\mathcal{N} = V_1 \oplus \cdots \oplus V_r$ be a Carnot algebra.

1. If $r_1(\mathcal{N}) = 1$, then $[W_1, V_i] = 0$ for all $i \geq 2$;
2. If $r_1(\mathcal{N}) \geq 2$ and $r_1, C(\mathcal{N}) \leq 1$, then $[\hat{W}_1, V_i] = 0$ for all $i \geq 2$.

**Proof.** (1) holds since $W_1$ is spanned by rank one elements and $[X, V_i] = 0$, $i \geq 2$ for $X \in V_1$ with $\text{rank}(X) = 1$.

(2) Lemma 3.5 implies $r_{1, C}(\mathcal{N}) = 1$. Hence $W_{1, C} \subset V_1 \otimes C$ is spanned by rank one elements. So $[W_{1, C}, V_i \otimes C] = 0$ for all $i \geq 2$. It follows that $[W_{1, C}, V_i] = 0$ for all $i \geq 2$. By definition, $\hat{W}_1$ is the set of real parts of elements in $W_{1, C}$. Hence (2) holds.

Notice that Theorem 1.1 is equivalent to the following:

**Theorem 5.8.** Let $\mathcal{N}$ be a non-rigid Carnot algebra. If $\mathcal{N}$ is not one of the following three classes of algebras, then it is quasisymmetrically rigid:

1. Euclidean algebras;
2. Heisenberg product algebras;
3. complex Heisenberg product algebras.

**Proof.** By Theorem 4.8 $\mathcal{N}$ has a reducible first layer. If $\mathcal{N}$ is 2-step, then Theorem 2.2 implies $\mathcal{N}$ is quasisymmetrically rigid. From now on we shall assume $\mathcal{N}$ is $r$-step with $r \geq 3$. If $r_1(\mathcal{N}) = 0$, then the claim follows from the main result in [X3]. Assume $r_1(\mathcal{N}) = 1$. Then Corollary 3.4 implies $W_1$ is a non-trivial proper subspace of $V_1$ invariant under the action of $\text{Aut}_g(\mathcal{N})$. If $W_1$ is abelian, then the claim follows from Lemma 5.7(1) and Theorem 5.5. If $W_1$ is not abelian, then the claim follows from Lemma 5.1.

Finally we assume $r_1(\mathcal{N}) \geq 2$ and $r_{1, C}(\mathcal{N}) \leq 1$. Then Corollary 3.7 implies $\hat{W}_1$ is a non-trivial proper subspace of $V_1$ invariant under the action of $\text{Aut}_g(\mathcal{N})$. If $\hat{W}_1$ is abelian, then the claim follows from Lemma 5.7(2) and Theorem 5.5. If $W_1$ is not abelian, then the claim follows from Lemma 5.2.

6 Examples of biLipschitz maps on rank one groups

In this section we construct an infinite dimensional space of biLipschitz maps on Carnot groups whose Lie algebras have rank one elements in the first layer. We remark that Ottazzi [O] has previously proved that under the above assumption the space of smooth contact vector fields is infinite dimensional. The main point here is to construct explicit maps and also provide lots of non-smooth biLipschitz maps.

There are several different constructions that yield rank one Carnot algebras, see Section 4.2 of [X2].
We shall need a result from [X1]. The n-step \((n \geq 2)\) model Filiform algebra \(F^n\) is an \((n + 1)\)-dimensional Lie algebra. It has a basis \(\{e_1, e_2, \ldots, e_n, e_{n+1}\}\) and the only non-trivial bracket relations are \([e_1, e_j] = e_{j+1}\) for \(2 \leq j \leq n\). The Lie algebra \(F^n\) admits a direct sum decomposition of vector subspaces \(F^n = V_1 \oplus \cdots \oplus V_n\), where \(V_1\) is the linear subspace spanned by \(e_1, e_2\), and \(V_j (2 \leq j \leq n)\) is the linear subspace spanned by \(e_{j+1}\). It is easy to check that \([V_1, V_j] = V_{j+1}\) for \(1 \leq j \leq n\), where \(V_{n+1} = \{0\}\). Hence \(F^n\) is a graded Lie algebra. The connected and simply connected Lie group with Lie algebra \(F^n\) will be denoted by \(F^n\) and is called the n-step model Filiform group.

Let \(h : \mathbb{R} \to \mathbb{R}\) be a L-Lipschitz function for some \(L \geq 1\). Set \(h_2 = h^2\) and define \(h_j : \mathbb{R} \to \mathbb{R}, 3 \leq j \leq n + 1,\) inductively as follows:

\[
h_j(x) = -\int_0^x h_{j-1}(s) ds.
\]

Define \(G_h : F^n \to F^n\) by:

\[
G_h(p) = p * \sum_{j=2}^{n+1} (h_j(x_1)) e_j
\]

for \(p = \sum_{j=1}^{n+1} x_j e_j \in F_n = F_n\). It was shown in Section 3.4 of [X1] that \(G_h\) is \(L'\)-biLipschitz, where \(L'\) is a constant depending only on \(L\).

We also need the following Lemma of Ottazzi:

**Lemma 6.1.** (Lemma 3 in [O]) Let \(N = V_1 \oplus \cdots \oplus V_r\) be a Carnot algebra. Suppose that there exists \(e_2 \in V_1\) such that \(\text{rank}(e_2) = 1\). Then \(N\) can be written as

\[
N = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \cdots \oplus \mathbb{R}e_{n+1} \oplus K,
\]

a vector space direct sum, where \(e_1 \in V_1\) and

\[
[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n,
\]

\[
e_{n+1} \in C(N), \quad \text{center of } N.
\]

Moreover,

\[
N_0 = \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_{n+1} \oplus K
\]

is an ideal and

\[
I = \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_{n+1}
\]

is abelian. Finally, \(V_1\) has a basis \(\{e_1, e_2, U_1, \cdots, U_s\}\) so that

\[
[e_j, U_i] = 0, \quad \forall j = 2, \cdots, n + 1, \forall i = 1, \cdots, s.
\]

Let \(F = \langle e_1, e_2 \rangle = (\mathbb{R}e_1 \oplus \mathbb{R}e_2) \oplus \mathbb{R}e_3 \oplus \cdots \oplus \mathbb{R}e_{n+1}\) be the subalgebra of \(N\) generated by \(e_1\) and \(e_2\). Fix a vector space basis \(B\) of \(N\) such that each element of \(B\) belongs to some \(V_k\) and \(e_i, U_j \in B\) for all \(i, j\). For any \(x \in N\), we let \(x_1\) be the coefficient of \(e_1\) when \(x\) is expressed as a linear combination of elements in \(B\).

24
Lemma 6.2. (1) \([V_i, \mathcal{I}] = 0\) for all \(i \geq 2\);
(2) For any \(x \in N\) and any \(y \in \mathcal{I}\), we have \([x, y] = x_1[e_1, y]\); in particular, \(\mathcal{I}\) is an ideal of \(N\).

Proof. (1) Notice that \(W := \mathbb{R}e_2\) satisfies the assumption of Lemma 5.4. Now the claim follows from (5.3) for \(\tilde{x}_1 = e_1\) and \(u = e_2\).

(2) For any \(x \in N\) and any \(y \in \mathcal{I}\), the equality \([x, y] = x_1[e_1, y]\) follows from (1), (6.4) and the fact that \([e_2, e_j] = 0\) for all \(j \geq 2\). Then (6.2) implies that \(\mathcal{I}\) is an ideal of \(N\).

Here is the main result in this Section:

Proposition 6.3. Let \(N\) be a Carnot group with Lie algebra \(N = V_1 \oplus \cdots \oplus V_r\). Suppose there is some \(e_2 \in V_1\) with \(\text{rank}(e_2) = 1\). Let \(U_1, \ldots, U_s \in V_1\) and \(e_i \in V_{i-1}\), \(2 \leq i \leq n + 1\) be given by Lemma 6.1. Let \(B\) be a basis of \(N\) such that each element of \(B\) belongs to some \(V_k\) and \(e_i, U_j \in B\) for all \(i, j\). Let \(h : \mathbb{R} \to \mathbb{R}\) be a Lipschitz function, and \(h_j\) be as defined at the beginning of this section. Then the map \(F_h : N \to N\),

\[ F_h(x) = x \ast (\sum_{j=2}^{n+1} h_j(x_1)e_j) \]

is biLipschitz, where \(x_1\) is the coefficient of \(e_1\) when \(x\) is written as a linear combination of elements of \(B\).

Proof. Let each \(V_i\) be equipped with the inner product such that \(V_i \cap B\) is an orthonormal basis. Let \(d\) be the corresponding homogeneous distance on \(N = N(\text{see the last paragraph of Section 2.1})\).

Let \(x, \tilde{x} \in N = N\) be two arbitrary points. We need to show that \(d(x, \tilde{x})\) and \(d(F_h(x), F_h(\tilde{x}))\) are comparable. Denote \(h_j = h_j(x_1)\) and \(\tilde{h}_j = h_j(\tilde{x}_1)\). Set \(y = \sum_{j=2}^{n+1} h_j e_j\) and \(\tilde{y} = \sum_{j=2}^{n+1} \tilde{h}_j e_j\). Then \(F_h(x) = x \ast y\) and \(F_h(\tilde{x}) = \tilde{x} \ast \tilde{y}\). We have

\[ d(x, \tilde{x}) = ||(-x) \ast \tilde{x}|| \]

and

\[ d(F_h(x), F_h(\tilde{x})) = d(x \ast y, \tilde{x} \ast \tilde{y}) = ||(-y) \ast (-x) \ast \tilde{x} \ast \tilde{y}||. \]

By Lemma 6.2 (1), \([V_i, \mathcal{I}] = 0\) for all \(i \geq 2\). Notice \(y, \tilde{y} \in \mathcal{I}\). So Lemma 5.3 can be applied to obtain

\[ ((-x) \ast \tilde{x}) \ast \tilde{y} = ((-x) \ast \tilde{x}) + \tilde{y} + \frac{1}{2} ((-x) \ast \tilde{x}, \tilde{y}) + \sum_{j=2}^{r-1} c_j (ad (-x) \ast \tilde{x})^j \tilde{y}. \]

Since \(((-x) \ast \tilde{x})_1 = \tilde{x}_1 - x_1\), Lemma 6.2 (2) implies

\[ ((-x) \ast \tilde{x}) \ast \tilde{y} = (-x) \ast \tilde{x} + \tilde{y} + \frac{1}{2} (\tilde{x}_1 - x_1)[e_1, \tilde{y}] + \sum_{j=2}^{r-1} c_j (\tilde{x}_1 - x_1)^j (ad e_1)^j \tilde{y} \]

\[ = (-x) \ast \tilde{x} + \tilde{y} + \frac{1}{2} (\tilde{x}_1 - x_1)[e_1, \tilde{y}] + \sum_{j=2}^{n-1} c_j (\tilde{x}_1 - x_1)^j (ad e_1)^j \tilde{y}. \quad (6.5) \]
A similar argument applied to \((-y) \ast ((-x) \ast \tilde{x} \ast \tilde{y})\) yields
\[
(-y) \ast ((-x) \ast \tilde{x} \ast \tilde{y})
= (-y) + ((-x) \ast \tilde{x} \ast \tilde{y}) + \frac{1}{2}[-y, ((-x) \ast \tilde{x} \ast \tilde{y})] + \sum_{j=2}^{n-1} (-1)^j c_j(((-x) \ast \tilde{x} \ast \tilde{y})^j(-y)
= (-y) + ((-x) \ast \tilde{x} \ast \tilde{y}) + \frac{1}{2}[-y, (\tilde{x}_1 - x_1)e_1] + \sum_{j=2}^{n-1} (-1)^j c_j(\tilde{x}_1 - x_1)^j(ad e_1)^j(-y).
\]

Using (6.5) we obtain:
\[
(-F_h(x)) \ast F_h(\tilde{x}) = (-y) \ast ((-x) \ast \tilde{x} \ast \tilde{y}) = (-x) \ast \tilde{x} + Y, \tag{6.6}
\]
where
\[
Y = (\tilde{y} - y) + \frac{1}{2}(\tilde{x}_1 - x_1)[e_1, \tilde{y} + y] + \sum_{j=2}^{n-1} c_j(\tilde{x}_1 - x_1)^j(ad e_1)^j(\tilde{y} - (-1)^jy).
\]

Notice that (6.6) implies the coefficient of \(e_1\) in both \((-x) \ast \tilde{x}\) and \((-F_h(x)) \ast F_h(\tilde{x})\) are \(\tilde{x}_1 - x_1\). Hence
\[
|\tilde{x}_1 - x_1| \leq ||(-x) \ast \tilde{x}|| = d(x, \tilde{x}) \tag{6.7}
\]
and
\[
|\tilde{x}_1 - x_1| \leq d(F_h(x), F_h(\tilde{x})). \tag{6.8}
\]

We observe that the model Filiform group \(F^n\) satisfies the assumption of the Proposition. On \(F^n\) we shall also use the norm and homogeneous distance defined at the beginning of the proof. All the above calculations and estimates are valid for \(F^n\) as well. Notice that \(F_h(x_1 e_1) = G_h(x_1 e_1)\) and \(F_h(\tilde{x}_1 e_1) = G_h(\tilde{x}_1 e_1)\). Since \((-x_1 e_1) \ast \tilde{x}_1 e_1 = (\tilde{x}_1 - x_1)e_1\), (6.6) implies
\[
(-F_h(x_1 e_1)) \ast F_h(\tilde{x}_1 e_1) = (\tilde{x}_1 - x_1)e_1 + Y = (-G_h(x_1 e_1)) \ast G_h(\tilde{x}_1 e_1).
\]

Since \(G_h\) is L-biLipschitz for some \(L \geq 1\), the triangle inequality now implies
\[
||Y|| \leq ||(\tilde{x}_1 - x_1)e_1 + Y|| = d(G_h(x_1 e_1), G_h(\tilde{x}_1 e_1)) \leq L \cdot d(x_1 e_1, \tilde{x}_1 e_1)
= L \cdot |(-x_1 e_1) \ast \tilde{x}_1 e_1| = L \cdot |\tilde{x}_1 - x_1|. \tag{6.9}
\]

Now (6.6), triangle inequality, (6.9) and (6.7) imply
\[
d(F_h(x), F_h(\tilde{x})) = ||(-F_h(x)) \ast F_h(\tilde{x})||
= ||(-x) \ast \tilde{x} + Y|| \leq ||(-x) \ast \tilde{x}|| + ||Y|| \leq d(x, \tilde{x}) + L \cdot |\tilde{x}_1 - x_1|
\leq (L + 1) \cdot d(x, \tilde{x}).
\]

Similarly, using (6.8) instead of (6.7) we obtain
\[
d(x, \tilde{x}) = ||(-x) \ast \tilde{x}|| = ||(-F_h(x)) \ast F_h(\tilde{x}) - Y||
\leq ||(-F_h(x)) \ast F_h(\tilde{x})|| + ||Y||
\leq d(F_h(x), F_h(\tilde{x})) + L \cdot |\tilde{x}_1 - x_1|
\leq d(F_h(x), F_h(\tilde{x})) + L \cdot d(F_h(x), F_h(\tilde{x}))
= (L + 1)d(F_h(x), F_h(\tilde{x})).
\]

26
Hence $F_h$ is biLipschitz.

□

References


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