Rigidity of quasiisometries of H MN associated with non-diagonalizable derivation of the Heisenberg algebra

Xiangdong Xie

Abstract

We study quasiisometries of the homogeneous manifold with negative curvature associated with non-diagonalizable derivation of the Heisenberg algebra. We show that all quasiisometries are almost isometries. We prove this by finding all the quasisymmetric maps on the ideal boundary.

Keywords. rigidity, quasiisometry, quasisymmetric map, homogeneous manifold with negative curvature, Heisenberg group.

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1 Introduction

In this paper we shall study the self quasiisometries of a homogeneous manifold with negative curvature. We do this by studying the self quasisymmetric maps of the ideal boundary.

Let $\mathcal{H}$ be the Heisenberg Lie algebra with basis $e_1, e_2, e_3$ and the only nontrivial bracket relation $[e_1, e_2] = e_3$. We shall identify $\mathcal{H}$ with the Heisenberg group $H$ via the exponential map. The group law on $H = \mathcal{H}$ is given by:

$$(x_1 e_1 + y_1 e_2 + z_1 e_3) * (x_2 e_1 + y_2 e_2 + z_2 e_3) = (x_1 + x_2) e_1 + (y_1 + y_2) e_2 + (z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)) e_3.$$

Let $A : \mathcal{H} \to \mathcal{H}$ be the derivation of $\mathcal{H}$ whose matrix representation with respect to $e_1, e_2, e_3$ is given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$ 

Let $\mathbb{R}$ act on $H$ by $(t, v) \to e^{tA} v$ $(t \in \mathbb{R}, v \in H)$. We denote the corresponding semi-direct product by $S = H \times_{A} \mathbb{R}$. That is, $S = H \times \mathbb{R}$ as a smooth manifold, and the group operation is given by:

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\[(v, t) \cdot (w, s) = (v + (e^{tA}w), t + s)\]
for all \((v, t), (w, s) \in H \times \mathbb{R}\). The group \(S\) is a simply connected solvable Lie group.

We endow \(S\) with the left invariant Riemannian metric determined by taking the standard Euclidean metric at the identity of \(S = H \times \mathbb{R} = H^3\). It follows from [H] that \(S\) is Gromov hyperbolic and so has a well defined ideal boundary \(\partial S\). There is a so-called cone topology on \(\Sigma = S \cup \partial S\), in which \(\partial S\) is homeomorphic to the 3-dimensional sphere and \(\Sigma\) is homeomorphic to the closed 4-ball in the Euclidean space. Furthermore, there are also so called visual metrics on the ideal boundary \(\partial S\). Every quasiisometry \(f : S \to S\) induces a quasisymmetric map \(\partial f : \partial S \to \partial S\) of the ideal boundary (equipped with a visual metric), and the rigidity property of the quasiisometry \(f\) corresponds to the rigidity property of the boundary map \(\partial f\), see [BS] and [SX]. We shall study the rigidity property of self quasisymmetries of \(S\) by studying the rigidity property of self quasisymmetric maps of the ideal boundary.

We next describe the ideal boundary and the parabolic visual metric. For each \(v \in H\), the map \(\gamma_v : \mathbb{R} \to S\), \(\gamma_v(t) = (v, t)\) is a geodesic. We call such a geodesic a vertical geodesic. It can be checked that all vertical geodesics are asymptotic as \(t \to +\infty\). Hence they define a point \(\xi_0\) in the ideal boundary \(\partial S\). Each geodesic ray in \(S\) is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. The upward oriented geodesics are asymptotic to \(\xi_0\) and the downward oriented vertical geodesics are in 1-to-1 correspondence with \(H\). Hence \(\partial S \setminus \{\xi_0\}\) can be naturally identified with \(H\).

For any proper Gromov hyperbolic geodesic space \(X\) and any \(\xi \in \partial X\), there are so-called parabolic visual (quasi)metrics on \(\partial X \setminus \{\xi\}\), see [BK], [Ha] or [HP]. In our case, a parabolic visual quasimetric \(D\) on \(\partial S \setminus \{\xi_0\}\) is given by:

\[
D(p, q) = \max \left\{ |y_2 - y_1|, |(x_2 - x_1) - (y_2 - y_1) \ln |y_2 - y_1||, |z_2 - z_1 + \frac{1}{2} (x_2 y_1 - x_1 y_2)|^2 \right\},
\]
for all \(p = x_1 e_1 + y_1 e_2 + z_1 e_3\), \(q = x_2 e_1 + y_2 e_2 + z_2 e_3 \in H\), where \(0 \ln 0\) is understood to be 0.

We remark that \(D\) is not a metric on \(\mathbb{R}^3\), but merely a quasimetric. This fact causes technical difficulties in the proof. Recall that a quasimetric \(\rho\) on a set \(A\) is a function \(\rho : A \times A \to \mathbb{R}\) satisfying the following three conditions:

1. \(\rho(x, y) = \rho(y, x)\) for all \(x, y \in A\);
2. \(\rho(x, y) \geq 0\) for all \(x, y \in A\) and \(\rho(x, y) = 0\) if and only if \(x = y\);
3. there is some \(M \geq 1\) such that \(\rho(x, z) \leq M(\rho(x, y) + \rho(y, z))\) for all \(x, y, z \in A\).

For each \(M \geq 1\), there exists a constant \(c_0 > 0\) such that \(\rho^\prime\) is biLipschitz equivalent to a metric for any quasimetric \(\rho\) with constant \(M\) and any \(0 < \epsilon \leq c_0\), see Proposition 14.5. in [Hn].

Let \(\eta : [0, \infty) \to [0, \infty)\) be a homeomorphism. A bijection \(F : X \to Y\) between two quasimetric spaces is \(\eta\)-quasisymmetric if for all distinct triples \(x, y, z \in X\), we have

\[
\frac{d(F(x), F(y))}{d(F(x), F(z))} \leq \eta \left( \frac{d(x, y)}{d(x, z)} \right).
\]

A map \(F : X \to Y\) is quasisymmetric if it is \(\eta\)-quasisymmetric for some \(\eta\).

The following is the main result of the paper.
Theorem 1.1. Every quasisymmetric map $F : (\mathcal{H}, D) \to (\mathcal{H}, D)$ is biLipschitz. Furthermore, a bijection $F : (\mathcal{H}, D) \to (\mathcal{H}, D)$ is a quasisymmetric map if and only if it is a composition of the following types of maps:

1. left translations;
2. The map $R_\pi : \mathcal{H} \to \mathcal{H}$ given by $R_\pi(xe_1 + ye_2 + ze_3) = -xe_1 - ye_2 + ze_3$;
3. automorphisms $\lambda_t = e^{tA}$ generated by the derivation $A$;
4. maps of the form $F_c(ye_2 * (xe_1 + ze_3)) = ye_2 * [(x + c(y))e_1 + (z + \int_0^y c(s)ds)e_3]$, where $c : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map.

Theorem 1.1 is the Heisenberg group counterpart for the main result in [X1]. In that paper, all quasisymmetric maps on the ideal boundary of $\mathbb{R}^2 \rtimes J\mathbb{R}$ was identified, where

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and $\mathbb{R}$ acts on $\mathbb{R}^2$ by $(t, v) \to e^{tJ}v$.

Theorem 1.1 has consequences for the rigidity of quasiisometries. Let $L \geq 1$ and $C \geq 0$. A (not necessarily continuous) map $f : X \to Y$ between two metric spaces is an $(L, A)$-quasiisometry if:

1. $d(x_1, x_2)/L - C \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + C$ for all $x_1, x_2 \in X$;
2. for any $y \in Y$, there is some $x \in X$ with $d(f(x), y) \leq C$.

In the case $L = 1$, we call $f$ an almost isometry.

Corollary 1.2. Every self quasiisometry of $S$ is an almost isometry.

Notice that an almost isometry is not necessarily a finite distance away from an isometry.

The theme of the paper is rigidity of quasiisometries of homogeneous manifolds with negative curvature (HMNs). Conjecturally all self quasiisometries of a HMN must be almost isometries unless the HMN is biLipschitz to a real or complex hyperbolic space. By [H] every HMN is isometric to a solvable Lie group $S$ with a left invariant Riemannian metric, and the solvable Lie group $S$ has the form $S = N \rtimes \mathbb{R}$, where $N$ is a simply connected nilpotent Lie group and $\mathbb{R}$ acts on $N$ by expanding automorphisms (for $t > 0$). The case $N = \mathbb{R}^n$ was solved in [X2]. The case where $N$ is a Heisenberg group and the derivation is diagonalizable was solved in [X3]. This paper is the first step in treating the non-diagonalizable derivations on Heisenberg groups. The general case still remains open.

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2 Parabolic visual quasimetrics on the ideal boundary

In this section, we will define two different parabolic visual quasimetrics on the ideal boundary, and find an explicit formula for one of them. The two quasimetrics are biLipschitz equivalent with each other.

Let $A$ and $S$ be as in the Introduction. Recall that all vertical geodesics $\gamma_v$ ($v \in \mathcal{H}$) are asymptotic as $t \to +\infty$ and so they define a point $\xi_0$ in the ideal boundary $\partial S$. Furthermore, $\partial S \setminus \{\xi_0\}$ can be naturally identified with $\mathcal{H}$. The subsets $\mathcal{H} \times \{t\} \subset \mathcal{H} \times \mathbb{R} = S$ ($t \in \mathbb{R}$) are horospheres centered at $\xi_0$.

We next define two parabolic visual quasimetrics on $\partial S \setminus \{\xi_0\} = \mathcal{H}$. Given $v, w \in \mathcal{H}$, the parabolic visual quasimetric $D_1(v, w)$ is defined as follows: $D_1(v, w) = e^t$, where $t$ is the smallest real number such that at height $t$ the two vertical geodesics $\gamma_v$ and $\gamma_w$ are at distance one apart in the horosphere $\mathcal{H} \times \{t\}$.

We now introduce a parabolic visual quasimetric $D$ which admits an explicit formula and is also biLipschitz equivalent with $D_1$. Define a norm on $\mathcal{H}$ by:

$$|xe_1 + ye_2 + ze_3|_A := \max\{|y|, |x - y \ln |y||, |z|^{\frac{1}{2}}\}.$$ 

Now define $D$ by: $D(p, q) = |(-p) * q|_A$. Let $p = x_1e_1 + y_1e_2 + z_1e_3$, $q = x_2e_1 + y_2e_2 + z_2e_3 \in \mathcal{H}$. Since $(-p) * q = (x_2 - x_1)e_1 + (y_2 - y_1)e_2 + ((z_2 - z_1) + \frac{1}{2}(x_2y_1 - x_1y_2))e_3$,

we have

$$D(p, q) = \max\{|y_2 - y_1|, |(x_2 - x_1) - (y_2 - y_1) \ln |y_2 - y_1||, |z_2 - z_1 + \frac{1}{2}(x_2y_1 - x_1y_2)|^{\frac{1}{2}}\},$$

where $0 \ln 0$ is understood to be $0$. At this point $D$ is just a function. We shall see that $D$ is indeed a quasimetric.

Let $g = (xe_1 + ye_2 + ze_3, t) \in \mathcal{H} \times \mathbb{R} = S$ and denote by $L_g : S \to S$ the left translation by $g$. We calculate

$$L_g(x'e_1 + y'e_2 + z'e_3, t') = ([x + e^t(x' + ty')]e_1 + [y + e^t'y']e_2 + [z + e^t z'] + \frac{1}{2} e^t(xy' - x'y - tyy')|e_3, t + t').$$

We see that $L_g$ maps vertical geodesics to vertical geodesics. It follows that $L_g$ induces a map $T_g : \mathcal{H} \to \mathcal{H}$,

$$T_g(x'e_1 + y'e_2 + z'e_3) = [x + e^t(x' + ty')]e_1 + [y + e^t'y']e_2 + [z + e^t z'] + \frac{1}{2} e^t(xy' - x'y - tyy')|e_3.$$

Since $L_g$ is an isometry of $S$ and it translates by $t$ in the vertical direction, the definition of the quasimetric $D_1$ shows that

$$D_1(T_g(p), T_g(q)) = e^t D_1(p, q)$$

for all $p, q \in \mathcal{H}$. In other words, $T_g$ is a similarity of $(\mathcal{H}, D_1)$ with similarity constant $e^t$. When $t = 0$, $T_g$ is simply a left translation on $\mathcal{H}$ and it is an isometry with respect to $D_1$. When $g = (0, t) \in \mathcal{H} \times \mathbb{R} = S$, we have $T_g = \lambda_t = e^{tA}$.
A direct calculation (using the formula for $D$) shows that $T_g$ is a similarity with respect to $D$ as well. In particular, left translations are isometries with respect to $D$ and $\lambda_t = e^{tA}$ is a similarity of $(H, D)$ with similarity constant $e^t$. It follows that $D_1$ and $D$ are biLipschitz equivalent. It also implies that $D$ is a quasimetric since $D_1$ is.

For later use (Section 5), we notice that the map $R_{\pi} : H \to H$ defined by

$$R_{\pi}(xe_1 + ye_2 + ze_3) = -xe_1 - ye_2 + ze_3,$$

is an isometry with respect to $D$.

### 3 Quasisymmetric maps preserve a foliation

In this section we prove that every self quasisymmetric map of $(H, D)$ preserves a foliation. The foliation consists of the left cosets of the $x$-axis. Notice that the $x$-axis is a subgroup of $H$. The first part of the proof is similar to Bruce Kleiner’s proof in the case of $\mathbb{R}^2 \ltimes J \mathbb{R}$; See [X1], Section 3.

We first recall some basics about $Q$-variation.

**Definition 3.1.** Let $(X, \rho)$ be a quasimetric space and $K \geq 1$. A subset $E \subset X$ is called a $K$-quasi-ball if there is some $x \in X$ and some $r > 0$ such that $B(x, r) \subset E \subset B(x, Kr)$. Here $B(x, r) = \{y \in X : \rho(y, x) < r\}$.

The following notion is key to the proof.

**Definition 3.2.** (Kleiner) Let $Q \geq 1$. Let $u : X \to \mathbb{R}$ be a function (not necessarily continuous) defined on a quasimetric space, and let $P$ be a collection of subsets of $X$. The $Q$-variation of $u$ over $P$, denoted $V_Q(u, P)$, is the quantity

$$\sum_{P \in P} [\text{osc}(u|_P)]^Q,$$

where $\text{osc}(u|_P)$ denotes the oscillation (sup minus inf) of the restriction of $u$ to the subset $P \subset X$. The $Q$-variation $V_Q(u)$ of $u$ is $\sup\{V_Q(u, P)\}$ where $P$ ranges over all disjoint collections of balls in $X$. For $K \geq 1$, the $(Q, K)$-variation $V_{Q,K}(u)$ of $u$ is $\sup\{V_Q(u, P)\}$ where $P$ ranges over all disjoint collections of $K$-quasi-balls in $X$.

The following result says that $Q$-variation is invariant under quasisymmetric maps in a certain sense.

**Lemma 3.1.** (Lemma 3.1, [X1]) Let $F : X \to Y$ be an $\eta$-quasisymmetric map between two quasimetric spaces. Then for every function $u : X \to \mathbb{R}$ we have $V_{Q,K}(u) \leq V_{Q,\eta(K)}(u \circ F^{-1})$.

By the discussion in Section 2, for each $g \in S$, the map $T_g : H \to H$ is a similarity with respect to $D$. Hence, the images of the unit cube

$$C = \{(xe_1 + ye_2 + ze_3 \in H : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$
under the action of $S$ on $\mathcal{H}$ are $K_0$-quasi-balls in $(\mathcal{H}, D)$ for some fixed $K_0 \geq 1$.

For any subset $S \subset \mathcal{H} = \mathbb{R}^3$, we use $m(S)$ to denote the Lebesgue measure of $S$. Recall that Lebesgue measure on $\mathcal{H}$ is a Haar measure on $\mathcal{H} = H$. In particular, it is invariant under left translations. Also notice that $m(\lambda_t(S)) = e^{4t}m(S)$ since $\lambda_t : \mathcal{H} \to \mathcal{H}$ has constant Jacobian determinant $e^{4t}$.

**Lemma 3.2.** The function $u_0 : \mathcal{H} \to \mathbb{R}$, $u_0(xe_1 + ye_2 + ze_3) = y$, has locally finite $(4, K)$-variation for any $K \geq 1$.

**Proof.** Let $U \subset \mathcal{H} = \mathbb{R}^3$ be any bounded open subset, and $K \geq 1$ a fixed constant. Let $\mathcal{P}$ be a collection of disjoint $K$-quasi-balls contained in $U$ and $E \in \mathcal{P}$. Then there exist $p \in \mathcal{H}$ and $r > 0$ such that $B(p, r) \subset E \subset B(p, Kr)$. By the discussion above, all sets of the form $T_q(C)$ are $K_0$-quasi-balls. Hence there exist $g_1, g_2 \in G_A$ such that $B(p, r/K_0) \subset T_{g_1}(C) \subset B(p, r)$ and $B(p, Kr) \subset T_{g_2}(C) \subset B(p, K_0 Kr)$. It follows that $\text{osc}(u_0|_E) \leq \text{osc}(u_0|_{T_{g_2}(C)})$. Notice that

$$[\text{osc}(u_0|_{T_{g_2}(C)})]^4 = m(T_{g_2}(C)) = (K_0 K)^4 m(T_{g_1}(C)) \leq (K_0 K)^4 m(E).$$

It follows that the $(4, K)$-variation of $u_0|_U$ is bounded from above by $(K_0 K)^4 m(U)$.

\[\Box\]

Denote by $X = \mathbb{R}e_1$ the $x$-axis and $Y = \mathbb{R}e_2$ the $y$-axis. Notice that both are connected subgroups of $\mathcal{H}$.

**Lemma 3.3.** Let $U \subset \mathcal{H}$ be an open subset. If $u : U \to \mathbb{R}$ is a continuous function which is not constant along some left coset of $X$ in $U$, then $V_{4, K_0}(u) = \infty$.

**Proof.** Since $u$ is continuous and is not constant along a left coset of $X$, after pre-composing with a left translation and $\lambda_{t_0}$ for some $t_0$ and post-composing with an affine function, we may assume the following: there exists some $a > 0$ such that $U$ contains the rectangular box

$$B = \{xe_1 + ye_2 + ze_3 : 0 \leq x \leq 1, 0 \leq y, z \leq a\};$$

furthermore, $u < 0$ on the set

$$F_0 = \{ye_2 + ze_3 : 0 \leq y, z \leq a\}$$

and $u > 1$ on the set

$$F_1 = \{e_1 + ye_2 + ze_3 : 0 \leq y, z \leq a\}.$$

Denote $Q_2 = \{me_2 : m \in \mathbb{Z}\}$ and $Q_{1, 3} = \{me_1 + ne_3 : m, n \in \mathbb{Z}\}$. Set

$$S_0 = \bigcup_{Q \in Q_2} q \ast C.$$

Notice that $Y \subset S_0$, and the Hausdorff distance $c_1 := HD(S_0, Y)$ is finite. Also notice that $\{q \ast S_0 : q \in Q_{1, 3}\}$ form a tessellation of $\mathcal{H}$ by translates of $S_0$. It follows that for any $t \in \mathbb{R}$, $\{\lambda_t(q \ast S_0) : q \in Q_{1, 3}\}$ form a tessellation of $\mathcal{H}$ by translates $\lambda_t(q) \ast \lambda_t(S_0) = \lambda_t(q \ast S_0)$ of $\lambda_t(S_0)$.
Notice that $\lambda_t = e^{tA}$ has the following matrix representation with respect to $e_1, e_2, e_3$:

$$A = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$ 

It follows that $\lambda_t(Y)$ is a line in $\mathcal{H}$ and for $t << 0$, it is almost parallel to the $x$-axis $X$. Since $HD(\lambda_t(S_0), \lambda_t(Y)) = ct e^t$, for $t << 0$, $\lambda_t(S_0)$ is almost parallel to $X$. For $i = 1, 2, 3$, let $p_i : \mathcal{H} \to \mathbb{R}$ be defined by $p_i(x_1e_1 + x_2e_2 + x_3e_3) = x_i$. Then $p_1(\lambda_t(C))$ is a closed interval with length $(|t| + 1)e^t$, $p_2(\lambda_t(C))$ has length $e^t$, and $p_3(\lambda_t(C))$ has length $e^{2t}$. Geometrically, $\lambda_t(C)$ is a thin slab roughly in the $x$-direction, and its projection on the $x$-axis has size roughly $|t|e^t$.

Now consider a translate $\lambda_t(q \ast S_0)$ of $\lambda_t(S_0)$ that connects $F_0$ and $F_1$. Set

$$\mathcal{I}_q = \{ \lambda_t(q \ast p \ast C) : p \in Q_2 \text{ and } \lambda_t(q \ast p \ast C) \subset B \}.$$ 

By the above estimate on the size of $p_1(\lambda_t(C))$, the cardinality of $\mathcal{I}_q$ is approximately $(|t|e^t)^{-1}$. By Jensen’s inequality, the $Q$-variation of $u$ over $\mathcal{I}_q$ is at least the $Q$-variation when the oscillations of $u$ on the members of $\mathcal{I}_q$ are all equal. That is, the $Q$-variation over $\mathcal{I}_q$ will be minimal when the oscillation on each member of $\mathcal{I}_q$ is $|\mathcal{I}_q|^{-1} \approx |t|e^t$. On the other hand, as observed above, $\{ \lambda_t(q \ast S_0) : q \in Q_{1,3} \}$ form a tessellation of $\mathcal{H}$ by translates of $\lambda_t(S_0)$. Also recall that for $t << 0$, $\lambda_t(S_0)$ is almost parallel to the $x$-axis. It follows that there is a finite subset $\tilde{Q}_{1,3} \subset Q_{1,3}$ such that for each $q \in \tilde{Q}_{1,3}$, $\lambda_t(q \ast S_0)$ connects $F_0$ and $F_1$, and $B' = B \cap (\cup q \in \tilde{Q}_{1,3} \lambda_t(q \ast S_0))$ has measure at least half of $m(B)$. Set $\mathcal{I} = \bigcup_{q \in \tilde{Q}_{1,3}} \mathcal{I}_q$.

Since $m(\lambda_t(C)) = e^{4t}$, the cardinality of $\mathcal{I}$ is at least $\frac{\alpha^2}{2e^{4t}}$. Hence the $Q$-variation of $u$ over $\mathcal{I}$ is at least

$$\frac{|t|e^t}{e^{4t}} \cdot \frac{\alpha^2/2}{e^{4t}} = \frac{1}{2} \alpha^2 t^4,$$

which goes to $\infty$ as $t \to -\infty$. Hence $V_{4,K_0}(u) = \infty$.

Let $H_1 = \mathbb{R}e_1 \oplus \mathbb{R}e_3$. Notice that $H_1$ is a connected subgroup of $\mathcal{H}$.

**Lemma 3.4.** For each left coset $L$ of $X$, $F(L)$ lies in a left coset of $H_1$.

**Proof.** Suppose that the claim in the lemma is false. Then there are two points $p, q \in L$ such that $F(p)$ and $F(q)$ lie in distinct left cosets of $H_1$. Then $u_0 \circ F$ is not constant along $L$. By Lemma 3.3, $V_{4,K_0}(u_0 \circ F) = \infty$. On the other hand, applying Lemma 3.1 to the function $u_0 \circ F : \mathcal{H} \to \mathbb{R}$ and $F : \mathcal{H} \to \mathcal{H}$, we obtain $V_{4,H_0(K_0)}(u_0) \geq V_{4,K_0}(u_0 \circ F) = \infty$. This contradicts Lemma 3.2.

**Lemma 3.5.** $F$ maps each left coset of $X$ to a left coset of $X$.

**Proof.** Notice that $(H_1, D)$ is isometric to the metric space $(\mathbb{R}^2, \rho)$, where

$$\rho((x_1, z_1), (x_2, z_2)) = \max\{|x_2 - x_1|, |z_2 - z_1|^\frac{1}{2}\}.$$
The following was independently proved in [T], Section 15 and [K]: for any open subsets $U, V \subset \mathbb{R}^2$, and any quasisymmetric map $f : (U, \rho) \to (V, \rho)$, the map $f$ sends horizontal line segments in $U$ to horizontal line segments in $V$.

Let $L$ be a left coset of $X$. By Lemma 3.4, $F(L)$ lies in a left coset of $H_1$. After pre-composing and post-composing $F$ with left translations, we may assume $L = X$ and $F(X) \subset H_1$. Suppose $F(X)$ is not a left coset of $X$. Then there exist two points $p, q \in X$ such that $F(p)$ and $F(q)$ lie in different horizontal lines (that is, parallel to the $x$-axis in $H_1 = \mathbb{R}^2$) in $H_1$. Let $r \in X$ be a point between $p$ and $q$. Applying Lemma 3.4 to $F^{-1}$ and the left coset $L_1$ of $X$ passing through $F(r)$, we see that $F^{-1}(L_1)$ lies in a left coset of $H_1$. Since $r \in F^{-1}(L_1)$, we see that $F^{-1}(L_1)$ lies in $H_1$. As $r$ varies between $p$ and $q$, we see that $F^{-1}$ maps an open subset $U$ of $H_1$ into $H_1$. Notice that $F^{-1}|_U : U \to F^{-1}(U)$ is also quasisymmetric. Now the first paragraph implies that $F^{-1}|_U$ maps horizontal line segments to horizontal line segments. In particular, $F^{-1}$ maps an open interval of $L_1$ that contains $F(r)$ to a horizontal line $L_2$ in $H_1$. Since $r \in F^{-1}(L_1)$, we have $L_2 = X$. It follows that $F$ maps an open interval of $X$ containing $r$ to a horizontal line in $H_1$. Since this holds for all $r \in X$ between $p$ and $q$, $F([p, q])$ must be horizontal. So we get a contradiction.

\[\square\]

Lemma 3.5 implies that every quasisymmetric map of the ideal boundary $\partial S$ fixes the point $\xi_0$. It follows that $S$ is not quasisometric to any finitely generated group. For more details on these claims, see [SX], Section 6.

4 Quasisymmetric maps are biLipschitz

In this section we show that every quasisymmetric map of $(\mathcal{H}, D)$ is biLipschitz. This follows easily from Lemma 3.5 and the main result in [LX]. We also derive rigidity properties about self quasisymmetries of $S$ (Corollary 1.2).

Let $K \geq 1$ and $C > 0$. A bijection $F : X_1 \to X_2$ between two quasimetric spaces is called a $K$-quasisimilarity (with constant $C$) if

$$\frac{C}{K} d(x, y) \leq d(F(x), F(y)) \leq C K d(x, y)$$

for all $x, y \in X_1$. When $K = 1$, we say $F$ is a similarity. It is clear that a map is a quasisimilarity if and only if it is a biLipschitz map. The point of using the notion of quasisimilarity is that sometimes there is control on $K$ but not on $C$.

We recall the following definition.

**Definition 4.1.** Let $X$ be a quasimetric space and $\alpha \in (0, 1]$, $L \geq 1$. We say $X$ is an $(\alpha, L)$-fibered metric space if $X$ admits a partition into unbounded closed subsets $\{X_\lambda\}_{\lambda \in \Lambda}$ with the following properties:

1. (Fibers are snow-flake equivalent to geodesic spaces) For each $\lambda$, there exists a geodesic space $(\tilde{X}_\lambda, d)$ such that $X_\lambda$ is $L$-biLipschitz to $(\tilde{X}_\lambda, d^\alpha)$;
2. (Sublinear divergence) For any $\lambda_1, \lambda_2 \in \Lambda$, there exist a sequence of points $y_i \in X_{\lambda_2}$ such that $\frac{d(y_i, X_{\lambda_1})}{d(y_i, y_1)} \to 0$ as $i \to \infty$.
(3) (parallel fibers are non-isolated) For any \( \lambda \), there exists a sequence of fibers \( X_{\lambda} \) such that \( X_{\lambda} \) and \( X_{\lambda} \) converge to \( X_{\lambda} \) in the Hausdorff distance; here we say two fibers \( X_{\lambda} \) and \( X_{\lambda} \) are parallel if \( d(p, X_{\lambda}) = HD(X_{\lambda}, X_{\lambda'}) = d(q, X_{\lambda}) \) for any \( p \in X_{\lambda} \) and any \( q \in X_{\lambda} \); (4) (Positive distance between fibers) For any two distinct fibers \( L_{1}, \quad L_{2}, \quad d(L_{1}, L_{2}) := \inf \{d(p, q)|p \in L_{1}, \quad q \in L_{2}\} > 0 \).

**Theorem 4.1.** ([LX], Theorem 1.1) Let \( X, Y \) be \((\alpha, L)\)-fibered quasimetric spaces for some \( \alpha \in (0, 1] \) and \( L \geq 1 \). Suppose \( F : X \to Y \) is a \( \eta\)-quasisymmetric map that sends fibers of \( X \) homeomorphically onto fibers of \( Y \). Then \( F \) is a \( K\)-quasisimilarity, where \( K \) depends only on \( \eta, \alpha \) and \( L \).

**Proposition 4.2.** Every \( \eta\)-quasisymmetric map \( F : (H, D) \to (H, D) \) is a \( K\)-quasisimilarity, where \( K \) depends only on \( \eta \).

**Proof.** Let \( F : (H, D) \to (H, D) \) be an \( \eta\)-quasisymmetric map. By Lemma 3.5, \( F \) permutes the left cosets of \( X \). The Proposition shall follow from Theorem 4.1. Here the fibers are left cosets of \( X \). Since \( X \) is a connected subgroup of \( H \), Lemma 4.4 and Lemma 4.5 in [LX] imply that Conditions (3) and (4) in Definition 4.1 are satisfied. It remains to check the first two conditions.

Condition (1). Let \( p = x_{1}e_{1}, \quad q = x_{2}e_{1} \) be two points on \( X \). The formula for \( D \) yields \( D(p, q) = |x_{2} - x_{1}| \). Hence \((X, D)\) is isometric to the real line; in particular, it is a geodesic metric space.

Condition (2). Let \( p = x_{0}e_{1} + y_{0}e_{2} + z_{0}e_{3} \) be an arbitrary point. Consider the two left cosets \( X \) and \( p \times X \). Since 

\[
(-te_{1}) * p * (te_{1}) = x_{0}e_{1} + y_{0}e_{2} + (z_{0} - y_{0}t)e_{3},
\]

we have

\[
D(te_{1}, p * te_{1}) = D(0, (-te_{1}) * p * (te_{1})) = \max\{|y_{0}|, |x_{0} - y_{0} \ln |y_{0}|, |z_{0} - y_{0}t|^{\frac{1}{2}}\},
\]

which implies sublinear divergence.

All the conditions in Definition 4.1 are satisfied. The Proposition now follows from Theorem 4.1.

We next draw some consequences about self quasiisometries of \( S \).

Under the identification of \( S \) with \( H \times \mathbb{R} \), we view the map \( h : H \times \mathbb{R} \to \mathbb{R}, \quad h(p, t) = t \) as the height function. A quasiisometry \( f : S \to S \) is height-respecting if \( |h(f(p, t)) - t| \) is bounded independent of \((p, t) \in S \). By [D] Lemma 7 and [SX] Section 6, the following 3 conditions are equivalent:

1. \( f \) is an almost isometry;
2. \( f \) is height-respecting;
3. the boundary map \( \partial f : \partial S \to \partial S \) is biLipschitz.

Now the following Corollary follows from Proposition 4.2.

**Corollary 4.3.** All self quasiisometries of \( S \) are almost isometries and are height-respecting.
5 Characterization of quasisymmetric maps

In this Section we prove Theorem 1.1, which gives a complete description of all self quasisymmetric maps of \((H, D)\).

Let \(F : (H, D) \to (H, D)\) be a quasisymmetric map. By Proposition 4.2 \(F\) is \(M\)-biLipschitz for some \(M \geq 1\).

Lemma 5.1. \(F\) permutes the left cosets of \(H_1\).

Proof. Two left cosets \(L_1\) and \(L_2\) of \(X\) are parallel if \(D(p, L_2) = D(q, L_1)\) for any \(p \in L_1, q \in L_2\). The calculation at the end of proof of Proposition 4.2 shows that \(X\) and \(p \ast X\) are parallel if and only if \(p \in H_1\). It follows that two left cosets \(L_1\) and \(L_2\) of \(X\) are parallel if and only if they lie in the same left coset of \(H_1\). Since \(F\) is biLipschitz, it must map parallel cosets to parallel cosets. The Lemma follows.

Lemma 5.2. There exist maps \(g : \mathbb{R} \to \mathbb{R}, f : \mathbb{R}^3 \to \mathbb{R}\) and \(h : \mathbb{R}^2 \to \mathbb{R}\) such that

\[
F(ye_2 \ast (xe_1 + ze_3)) = g(y)e_2 \ast (f(x, y, z)e_1 + h(y, z)e_3).
\]

Furthermore, the following hold: (1) \(g\) is biLipschitz; (2) for each fixed \(y \in \mathbb{R}\), the map \(h(y, \cdot) : \mathbb{R} \to \mathbb{R}\) is biLipschitz; (3) for any fixed \(y, z \in \mathbb{R}\), the map \(f(\cdot, y, z) : \mathbb{R} \to \mathbb{R}\) is biLipschitz.

Proof. By Lemma 5.1, there exists a homeomorphism \(g : \mathbb{R} \to \mathbb{R}\) such that

\[
F(ye_2 \ast H_1) = g(y)e_2 \ast H_1.
\]

For each \(y \in \mathbb{R}\), there is some homeomorphism \(F_y : H_1 \to H_1\) such that

\[
F(ye_2 \ast p) = g(y)e_2 \ast F_y(p)
\]

for \(p \in H_1\). Since \(F\) is quasisymmetric, \(F_y : (H_1, D|_{H_1}) \to (H_1, D|_{H_1})\) is also quasisymmetric. As already observed in the proof of Lemma 3.5, \((H_1, D|_{H_1})\) is isometric to \((\mathbb{R}^2, \rho)\). So by the results cited there, \(F_y\) permutes the horizontal lines in \(H_1\). It follows that there is some homeomorphism \(h(y, \cdot) : \mathbb{R} \to \mathbb{R}\) such that \(F_y(ze_3 \ast X) = h(y, z)e_3 \ast X\). For each \(z \in \mathbb{R}\), then there is some homeomorphism \(f(\cdot, y, z) : \mathbb{R} \to \mathbb{R}\) such that \(F_y(xe_1 + ze_3) = f(x, y, z)e_1 + h(y, z)e_3\). Hence \(F\) has the required form.

Let \(ye_2 \ast H_1, ye_2 \ast H_1\) be two left cosets of \(H_1\). By using the formula for \(D\), it is easy to pick \(p \in ye_2 \ast H_1\) and \(q \in ye_2 \ast H_1\) such that \(D(p, q) = |y_2 - y_1|\). Since \(F\) is \(M\)-biLipschitz for some \(M \geq 1\), we have

\[
|g(y_2) - g(y_1)| \leq D(F(p), F(q)) \leq M \cdot D(p, q) = M \cdot |y_2 - y_1|.
\]

So \(g\) is Lipschitz. The same argument applied to \(F^{-1}\) implies that \(g^{-1}\) is also Lipschitz. Hence \(g\) is biLipschitz. A similar argument yields (2). Since by Proposition 4.2 \(F\) is biLipschitz, its restriction to any left coset \(L\) of \(X\) is also biLipschitz. On the other hand, \((L, D)\) is isometric to the real line. Hence (3) holds.
Lemma 5.3. Let $y_0 \in \mathbb{R}$ be such that $g'(y_0)$ exists, $z_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $\frac{\partial f}{\partial z}$ exists at $(x_0, y_0, z_0)$. Then $g'(y_0) = \frac{\partial f}{\partial z}(x_0, y_0, z_0)$.

Proof. Denote $p_0 = y_0e_2 \ast (x_0e_1 + z_0e_3)$. By replacing $F$ with $T_{-F(p_0)} \circ F \circ T_{p_0}$, we may assume $p_0 = F(p_0) = 0$. Here $T_p$ denotes the left translation by $p$. Since $g$ is biLipschitz, $g'(0) \neq 0$. By composing $F$ with the automorphism (which is a dilation with respect to $D$) $\lambda_t = e^{tA}$ for $t = -\ln |g'(y_0)|$ we may assume $g'(0) = 1$ or $-1$.

If $g'(0) = -1$, we further compose $F$ with the rotation $R_\pi : \mathcal{H} \rightarrow \mathcal{H}$, $R_\pi(xe_1 + ye_2 + ze_3) = -xe_1 - ye_2 + ze_3$. Hence we may assume $g'(0) = 1$. Denote $\lambda = \frac{\partial f}{\partial x}(0, 0, 0)$. We shall prove that $\lambda = 1$.

Since $\lambda_t$ is a similarity, the family of maps $\{F^t := \lambda_t \circ F \circ \lambda_{-t}| t \in \mathbb{R}\}$ consists of $M$-biLipschitz maps. Since $F^t(0) = 0$, Arzela-Ascoli Theorem implies that there is a sequence $t_i \rightarrow \infty$ such that $F^t_i$ converges uniformly on compact subsets towards an $M$-biLipschitz map $\bar{F} : (\mathcal{H}, D) \rightarrow (\mathcal{H}, D)$. Write

$$F^t(ye_2 \ast (xe_1 + ze_3)) = g^t(y)e_2 \ast (f^t(x, y, z)e_1 + h^t(y, z)e_3).$$

We notice that $g^t(y) = e^t \cdot g(e^{-t}y)$ and $f^t(x, 0, 0) = e^t f(e^{-t}x, 0, 0)$. Since the derivative $g'(0) = 1$ exists the maps $g^t(y) : \mathbb{R} \rightarrow \mathbb{R}$ converge (as $t \rightarrow \infty$) uniformly on compact subsets towards the identity map $y \rightarrow y$. Similarly, since $\frac{\partial f}{\partial x}(0, 0, 0)$ exist, the maps $f^t(\cdot, 0, 0) : \mathbb{R} \rightarrow \mathbb{R}$ converge (as $t \rightarrow \infty$) uniformly on compact subsets towards the map $x \rightarrow \lambda x$. Write $\bar{F}(ye_2 \ast (xe_1 + ze_3)) = \bar{g}(y)e_2 \ast (\bar{f}(x, y, z)e_1 + \bar{h}(y, z)e_3)$. The above discussion implies $\bar{g}(y) = y$ and $\bar{f}(x, 0, 0) = \lambda x$.

Fix some $x \in \mathbb{R}$ and a positive integer $n$. For $i = 0, \cdots, n$, let $p_i = y_i e_2 \ast (x_i e_1 + z_i e_3)$, where $x_i = x - \frac{i}{n} \ln n$, $y_i = \frac{i}{n}$ and $z_i = \frac{i}{n} x - \frac{\beta^3}{2n^2} \ln n$. Then $D(p_i, p_{i+1}) = 1/n$. Hence

$$|\bar{f}(x, y_i, z_i) - \bar{f}(x_{i+1}, y_{i+1}, z_{i+1}) - \frac{1}{n} \ln n| \leq D(\bar{f}(p_i), \bar{f}(p_{i+1})) \leq M \cdot \frac{1}{n}.$$ 

Adding up all these inequalities for $i = 0, \cdots, n - 1$ and using the triangle inequality we obtain

$$|\bar{f}(x_0, y_0, z_0) - \bar{f}(x_n, y_n, z_n) - \ln n| \leq M. \quad (5.1)$$

Set $q = (x - \ln n)e_1$. Notice that $D(p_n, q) = \sqrt{\ln n/2}$ for $n \geq 9$ and hence

$$|\bar{f}(x_n, y_n, z_n) - \bar{f}(x - \ln n, 0, 0)| \leq D(\bar{f}(p_n), \bar{f}(q)) \leq M \cdot \sqrt{\ln n/2}. \quad (5.2)$$

It follows from (5.1) and (5.2) that

$$|\bar{f}(x_0, y_0, z_0) - \bar{f}(x - \ln n, 0, 0) - \ln n| \leq M + M \cdot \sqrt{\ln n/2}.$$ 

Notice that $\bar{f}(x_0, y_0, z_0) = \bar{f}(x, 0, 0) = \lambda x$ and $\bar{f}(x - \ln n, 0, 0) = \lambda(x - \ln n)$. So we have $|\lambda - 1) \ln n| \leq M + M \cdot \sqrt{\ln n/2}$. Since this is true for all $n \geq 9$, we must have $\lambda = 1$.

\[\square\]

Lemma 5.4. There exist constants $a \neq 0$ and $b$ and also a function $c : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

1. $g(y) = ay + b$;
2. $f(x, y, z) = ax + c(y, z)$ for all $(x, y, z) \in \mathbb{R}^3$.  

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Proof. Let \( y \in \mathbb{R} \) be any point where \( g \) is differentiable. By Lemma 5.3, for any fixed \( z \in \mathbb{R} \), the biLipschitz map \( f(\cdot,y,z) : \mathbb{R} \to \mathbb{R} \) a.e. has derivative \( g'(y) \). It follows that \( f(\cdot,y,z) \) is an affine map; to be more precise, there is a constant \( c(y,z) \) depending only on \( y,z \) such that \( f(x,y,z) = g'(y)x + c(y,z) \) for all \( x \in \mathbb{R} \).

We claim that \( g'(y_1) = g'(y_2) \) holds for any two points \( y_1, y_2 \in \mathbb{R} \) at which \( g \) is differentiable. By the previous paragraph, \( f(x,y_1,0) = g'(y_1)x + c(y_1,0) \) and \( f(x,y_2,0) = g'(y_2)x + c(y_2,0) \). Notice that

\[
D(y_1e_2 \ast xe_1, y_2e_2 \ast xe_1) = \max\{|y_2 - y_1|, |(y_2 - y_1)\ln|y_2 - y_1||, |x(y_2 - y_1)|^{\frac{3}{2}}\}.
\]

So \( D(y_1e_2 \ast xe_1, y_2e_2 \ast xe_1) = |x(y_2 - y_1)|^{\frac{3}{2}} \) for sufficiently large \( x \). Since \( F \) is \( M \)-biLipschitz, we have

\[
D(F(y_1e_2 \ast xe_1), F(y_2e_2 \ast xe_1)) \leq M \cdot |x(y_2 - y_1)|^{\frac{3}{2}} \tag{5.3}
\]

for sufficiently large \( x \). On the other hand,

\[
F(y_1e_2 \ast xe_1) = g(y_1)e_2 \ast [(g'(y_1)x + c(y_1,0))e_1 + h(y_1,0)e_3]
\]

and

\[
F(y_2e_2 \ast xe_1) = g(y_2)e_2 \ast [(g'(y_2)x + c(y_2,0))e_1 + h(y_2,0)e_3].
\]

It follows that

\[
D(F(y_1e_2 \ast xe_1), F(y_2e_2 \ast xe_1)) \\
\geq |(g'(y_2)x + c(y_2,0)) - (g'(y_1)x + c(y_1,0)) - (g(y_2) - g(y_1))\ln|g(y_2) - g(y_1)||. \tag{5.4}
\]

Since (5.3) and (5.4) hold for all \( x \), we must have \( g'(y_1) = g'(y_2) \). Since \( g \) is biLipschitz, we see that \( g \) must be an affine function. Hence there exist constants \( a \neq 0 \) and \( b \in \mathbb{R} \) such that \( g(y) = ay + b \). The Lemma follows.

\[\square\]

**Lemma 5.5.** The function \( c(y,z) \) in Lemma 5.4 depends only on \( y \). Furthermore, \( c(y,z) = c(y) \) is a Lipschitz function of \( y \).

Proof. Let \( y_1, y_2, z_1 \in \mathbb{R} \) be arbitrary. For any \( x \in \mathbb{R} \), let \( x_2 = x_1 + (y_2 - y_1)\ln|y_2 - y_1| \) and

\[
z_2 = z_1 + x_1(y_2 - y_1) + \frac{1}{2}(y_2 - y_1)^2\ln|y_2 - y_1|.
\]

Denote \( p = y_1e_2 \ast (x_1e_1 + z_1e_3) \) and \( q = y_2e_2 \ast (x_2e_1 + z_2e_3) \). Then \( D(p,q) = |y_2 - y_1| \). Since \( F \) is \( M \)-biLipschitz, we have \( D(F(p),F(q)) \leq M \cdot D(p,q) = M \cdot |y_2 - y_1| \). On the other hand,

\[
F(p) = (ay_1 + b)e_2 \ast [(ax_1 + c(y_1,z_1))e_1 + h(y_1,z_1)e_3]
\]

and

\[
F(q) = (ay_2 + b)e_2 \ast [(ax_2 + c(y_2,z_2))e_1 + h(y_2,z_2)e_3].
\]

Hence

\[
M \cdot |y_2 - y_1| \geq D(F(p), F(q)) \\
\geq |ax_2 + c(y_2,z_2) - ax_1 - c(y_1,z_1) - a(y_2 - y_1)\ln|a(y_2 - y_1)|| \\
= |c(y_2,z_2) - c(y_1,z_1) - a\ln|a(y_2 - y_1)||.
\]

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Triangle inequality then implies

\[ |c(y_2, z_2) - c(y_1, z_1)| \leq (M + |a \ln |a||)|y_2 - y_1|. \] (5.5)

Notice that if \( y_1 \neq y_2 \), then \( z_2 \) can take on any real number for a suitable choice of \( x_1 \). It follows that (5.5) holds for all \( y_1 \neq y_2 \) and all \( z_1, z_2 \in \mathbb{R} \).

Now Let \( y \in \mathbb{R} \) be fixed and \( z_1, z_2 \in \mathbb{R} \). We need to show \( c(y, z_1) = c(y, z_2) \). By (5.5) the following holds for all \( y' \neq y \):

\[ |c(y, z_1) - c(y, z_2)| \leq |c(y, z_1) - c(y', z_1)| + |c(y', z_1) - c(y, z_2)| \leq 2(M + |a \ln |a||) \cdot |y' - y|. \]

Now we see \( c(y, z_1) = c(y, z_2) \) by letting \( y' \to y \). Finally, (5.5) implies that \( c(y, z) = c(y) \) is a Lipschitz function of \( y \).

\[ \square \]

**Completing the proof of Theorem 1.1.** The first statement is simply Proposition 4.2. For the second statement, we first show that all 4 types of maps listed in Theorem 1.1 are biLipschitz. Clearly the first 3 classes of maps are similarities. Now let \( F_c : \mathcal{H} \to \mathcal{H} \) be a map given by

\[ F_c(ye_2 * (xe_1 + ze_3)) = ye_2 * [(x + c(y))e_1 + (z + \int_0^y c(s)ds)e_3], \]

where \( c : \mathbb{R} \to \mathbb{R} \) is \( M \)-Lipschitz for some \( M \geq 0 \). One checks by direct calculation that \( F_c \) is Lipschitz, as follows. Let \( p = ye_2 * (xe_1 + ze_3), q = y'e_2 * (x'e_1 + z'e_3) \in \mathcal{H} \) be two arbitrary points. Then

\[ F_c(p) = ye_2 * [(x + c(y))e_1 + (z + \int_0^y c(s)ds)e_3] \]

and

\[ F_c(q) = y'e_2 * [(x' + c(y'))e_1 + (z' + \int_0^{y'} c(s)ds)e_3]. \]

Set \( \tau_1 = (x' - x) - (y' - y) \ln |y' - y| \) and \( \tau_2 = z' - z - \frac{1}{2}(y' - y)(x' + x) \). We have \( D(p, q) = \max\{ |y' - y|, |\tau_1|, |\tau_2|^{\frac{1}{2}} \} \) and

\[ D(F_c(p), F_c(q)) = \max \{ |(y' - y)|, |\tau_1 + [c(y') - c(y)]|, |\tau_2|^{\frac{1}{2}} \}, \]

where \( \tau_3 = \tau_2 + \int_0^{y'} c(s)ds - \frac{1}{2}(y' - y)(c(y') + c(y)) \). Clearly, \(|(y' - y)| \leq D(p, q)\). Since \( c \) is \( M \)-Lipschitz, we have

\[ |\tau_1 + [c(y') - c(y)]| \leq |\tau_1| + M \cdot |y' - y| \leq (M + 1)D(p, q). \]
We need to bound $\tau_3$ from above. We have

$$| \int_y^{y'} c(s)ds - \frac{1}{2}(y' - y)(c(y') + c(y))|$$

$$= | \int_y^{y'} c(s)ds - \frac{1}{2} \int_y^{y'} c(y)ds - \frac{1}{2} \int_y^{y'} c(y')ds|$$

$$=|\frac{1}{2} \int_y^{y'} [c(s) - c(y)]ds + \frac{1}{2} \int_y^{y'} [c(s) - c(y')]ds|$$

$$\leq \frac{1}{2} \int_y^{y'} M \cdot (s - y)ds + \frac{1}{2} \int_y^{y'} M \cdot (y' - s)ds$$

$$= \frac{1}{2} M(y' - y)^2.$$

It follows that

$$|\tau_3|^{\frac{1}{2}} \leq (|\tau_2| + \frac{1}{2} M(y' - y)^2)^{\frac{1}{2}}$$

$$\leq |\tau_2|^{\frac{1}{2}} + (\frac{M}{2} y' - y)^{\frac{1}{2}}$$

$$= |\tau_2|^{\frac{1}{2}} + (\frac{1}{2} M)^{\frac{1}{2}} \cdot |y' - y|$$

$$\leq (1 + (\frac{1}{2} M)^{\frac{1}{2}}) \cdot D(p, q).$$

Hence $F_c$ is Lipschitz.

Notice that $F_c^{-1} = F_{-c}$. Since $-c$ is also $M$-Lipschitz, we see that $F_c^{-1}$ is also Lipschitz. Hence $F_c$ is biLipschitz.

Next we suppose $F : (\mathcal{H}, D) \to (\mathcal{H}, D)$ is a quasisymmetric map and shall show that $F$ is a composition of the 4 types of maps. By Lemma 5.2, Lemma 5.4 and Lemma 5.5 $F$ has the form

$$F(ye_2 * (xe_1 + ze_3)) = (ay + b)e_2 * [(ax + c(y))e_1 + h(y, z)e_3],$$

where $a \neq 0$, $b$ are constants, and $c : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. After composing $F$ with a left translation, $\lambda_t$ for some $t$ and $R_\pi$ if necessary, we may assume $a = 1$ and $b = 0$. By further composing with $F_{-c}$ if necessary, we may assume that $c = 0$. Now $F$ has the form

$$F(ye_2 * (xe_1 + ze_3)) = ye_2 * [xe_1 + h(y, z)e_3].$$

We shall show that $h(y, z)$ has the form $h(y, z) = z + C$ for some constant $C$.

Let $p = ye_2 * ze_3$ and $q = ye_2 * ze_3$ with $|y' - y| < 1/e$. Then

$$D(p, q) = |(y' - y)\ln|y' - y||.$$

Also $F(p) = ye_2 * h(y, z)e_3$ and $F(q) = ye_2 * h(y', z)e_3$. So

$$D(F(p), F(q)) = \max\{|y' - y|, |(y' - y)\ln|y' - y||, |h(y', z) - h(y, z)|^{\frac{1}{2}}\}.$$
By Proposition 4.2, \( F \) is M-biLipschitz for some \( M \geq 1 \). It follows that 
\[
|h(y', z) - h(y, z)| \leq M \cdot |(y' - y) \ln |y' - y||.
\]
Hence, 
\[
\frac{|h(y', z) - h(y, z)|}{y' - y} \leq M^2 \cdot |y' - y|(\ln |y' - y|)^2.
\]
Since the right hand side goes to 0 as \( y' \to y \), we see that \( \frac{\partial h}{\partial y}(y, z) = 0 \) for all \((y, z)\). It follows that 
\[
h(y, z) = h(z) \text{ is a function of } z \text{ only}.
\]

Let \( y \neq y' \) and \( z, z' \) be arbitrary. Let \( x, x' \in \mathbb{R} \) be determined by the two equations 
\[
x' = x + (y' - y) \ln |y' - y| \quad \text{and} \quad z' = z + \frac{1}{2}(x' + x)(y' - y).
\]
Denote 
\[
p = ye_2 * (xe_1 + ze_3),
\]
\[
q = ye_2 * (x'e_1 + z'e_3).
\]
Then 
\[
D(p, q) = |y' - y|.
\]
We have 
\[
F(p) = ye_2 * (xe_1 + h(z)e_3) \quad \text{and} \quad F(q) = ye_2 * (x'e_1 + h(z')e_3).
\]
Hence 
\[
D(F(p), F(q)) = \max\{|y' - y|, 0, |h(z') - h(z) - \frac{1}{2}(x' + x)(y' - y)|^{\frac{1}{2}}\}
\] 
\[
= \max\{|y' - y|, 0, |h(z') - h(z) - (z' - z)|^{\frac{1}{2}}\}.
\]
It follows that 
\[
|h(z') - z' - (h(z) - z)|^{\frac{1}{2}} \leq D(F(p), F(q)) \leq M \cdot D(p, q) = M \cdot |y' - y|.
\]
Since the left hand side does not depend on \( y, y' \) and the inequality holds for all \( y \neq y' \), we must have 
\[
(h(z') - z') - (h(z) - z) = 0.
\]
Hence there is some constant \( C \) such that \( h(z) - z = C \) for all \( z \in \mathbb{R} \).

\[\square\]

**References**


[X2] X. Xie, *Large scale geometry of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$*, Geometry & Topology 18 (2014) 831–872.


Address:
Xiangdong Xie: Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, U.S.A. E-mail: xiex@bgsu.edu