BGSU MINI-COURSE ON BAKER-BEYNON DUALITY
A SUMMARY OF THE MAIN THEOREMS DISCUSSED, WITH REFERENCES

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Abstract. This document contains a rough summary of the main theorems discussed during the mini-course on Baker-Beynon Duality for unital lattice-ordered Abelian groups and MV-algebras held at Bowling Green State University from 8.VI.2009 through 12.VI.2009, with references to the original sources.

Bibliographic Note. The standard reference on MV-algebras is [10]. It develops all of the basic theory in careful detail. It does not cover the more advanced results, nor the more recent ones. In [10], Baker-Beynon duality for MV-algebras is not really dealt with, although much useful material needed for its development is to be found there. Regrettably, at the time of writing I know of no single source that adequately covers Baker-Beynon duality for MV-algebras or for unital lattice-ordered Abelian groups. Because MV-algebras were introduced as the algebraic counterpart of Łukasiewicz infinite-valued propositional logic, there are some books in mathematical logic that deal with MV-algebras; in this context, [16] is a standard reference.

There are several standard references for partially ordered and lattice-ordered groups. For those interested in doing research in the subject, it is definitely worthwhile to get acquainted with as many of them as possible. Let me mention at least [7], still a source of inspiration to this day; [13], with a nice early treatment of general partially ordered algebraic systems; [6], with the most complete treatment of the prime spectrum as a topological space to be found to date in handbooks on lattice-ordered groups; [1], which includes a treatment of groups of divisibility, along with a useful, annotated list of examples of lattice-ordered groups with special properties; [11], with a coverage that is both broad and deep, and with an authoritative treatment of varieties; [20], with several things not to be found elsewhere, such as the description of all total orders on free Abelian groups of finite rank in §6.3; [14], which includes both an account of order-preserving permutations by a leading expert, and a useful partial coverage of Baker-Beynon duality for the non-unital case; and finally [15], an important book that deals with the connection between *dimension groups*, a class of partially-ordered Abelian groups larger than the lattice-ordered ones, and C^**-algebras*, certain rings with additional structure that are investigated in functional analysis and mathematical physics.

Day 1 – Monday, 8.VI.2009. *Preliminaries.* We introduced MV-algebras and lattice-ordered Abelian groups with a distinguished (strong order) unit, *unital Abelian ℓ-groups* for short. As important examples, we introduced $(\mathbb{R}, 1)$, the totally ordered Abelian group of real numbers (under addition) with its natural order and

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unit 1, and $(\mathrm{C}(X), 1)$, the Abelian ℓ-group $\mathrm{C}(X)$ of all continuous functions from a compact Hausdorff space X to the real numbers, with unit the function constantly equal to 1, and operations defined pointwise from $(\mathbb{R}, 1)$. On the same note, we introduced the MV-algebra $\mathrm{C}_{[0,1]}(X)$ of $[0,1]$-valued continuous function on a compact Hausdorff space X, with operations define pointwise from the standard MV-algebra on the real unit interval $[0,1]$. The latter is the MV-algebra $([0,1], \oplus, \neg, 0)$ with *truncated addition* (or *Łukasiewicz disjunction*) $x \oplus y = \min\{x+y, 1\}$, *negation* $\neg x = 1 - x$, and *neutral element* 0. There is an underlying definable lattice structure on any MV-algebra given by $x \vee y = (x \odot \neg y) \oplus y$ — where $x \odot y = \neg(\neg x \oplus \neg y)$ is the *Łukasiewicz conjunction* — and $x \wedge y = \neg(\neg x \vee \neg y)$. Then 0 is the bottom element of the lattice, and $\neg 0$ (written 1) is the top one. Over $[0,1]$, with these definitions one obtains $x \odot y = \max\{0, x+y-1\}$, $x \wedge y = \min\{x, y\}$, and $x \vee y = \max\{x, y\}$.

The natural morphisms $h\colon (G,u) \to (H,w)$ between unital ℓ-groups are ℓ-homomorphisms $h\colon G \to H$ (i.e. functions that are both lattice and group homomorphisms) that preserve the units, that is, such that $h(u) = w$. We briefly discussed kernels of (unital) ℓ-homomorphisms between unital Abelian ℓ-groups, i.e. convex sublattice subgroups, known as *ℓ-ideals*. Similar considerations apply to MV-algebras – here, again, the natural morphisms are the homomorphisms, in the usual sense of functions preserving all operations.

The connection between MV-algebras and lattice-ordered groups is given by:

Theorem (Mundici's categorical equivalence [25])**.** *The category of MV-algebras* (*and their homomorphisms*) *and the category of unital Abelian ℓ-groups* (*and their unital ℓ-homomorphisms*) *are equivalent.*

In one direction, the functor traditionally known as Γ acts on unital Abelian ℓ-groups as follows. Given (G,u), $\Gamma(G,u)$ is the MV-algebraic structure on $[0,u] = \{g \in G \mid 0 \leqslant g \leqslant u\}$ given by the operations $x \oplus y = (x+y) \wedge u$ and $\neg x = u - x$, with bottom element 0. So, for example, $\Gamma(\mathbb{R}, 1) = [0,1]$ is the standard MV-algebra on the real unit interval, and $\Gamma(\mathrm{C}(X), 1) = \mathrm{C}_{[0,1]}(X)$. In the other direction, the action on MV-algebras of the functor adjoint to Γ is harder to describe. A full treatment, also covering algebraic structures that are more general than MV-algebras, was given by Nick Galatos in his mini-course.

Remark (Background in category theory)**.** Loosely speaking, the import of Mundici's result is that the mathematics of MV-algebras is essentially the same as the mathematics of unital Abelian ℓ-groups, up to the language adopted.[1] To make this statement precise, however – and indeed, to fully understand Mundici's theorem – one needs some background in category theory. For lack of time, I did not discuss the required notions in the mini-course. The same background knowledge is needed for Baker-Beynon duality, see Day 5 below, and is also illuminating in connection with free objects, see Day 4. Please see [21, 22] for the needed background. Approaching these references without previous exposure to category theory might prove challenging. It helps to bear in mind that what you really need to understand deeply is a single notion – adjoint functors. Though admittedly complex,

[1] *Caution.* We did point out that there are important differences too, especially when one is concerned with definability issues within fragments of predicate logic. An easy compactness argument shows that no family of formulæ of the predicate calculus can define the notion of strong order unit, whereas MV-algebras are by definition an equationally definable class of algebras.

adjointness truly is one of the central ideas in contemporary mathematics, *and is well worth of study whichever your field of interest is.* Once you understand adjoint functors, notions such as equivalent categories and free objects and categorical dualities will come easily as either special cases or by-products.

From this point onward in the minicourse, we adopted the language of unital Abelian ℓ-groups rather than that of MV-algebras; as just mentioned, this is for most purposes a matter of taste.

DAY 2 – TUESDAY, 9.VI.2009. *Weinberg's Theorem, and Chang's Completeness Theorem.* We discussed an elementary proof of the following

Theorem (Weinberg's Theorem [27]). $(\mathbb{Z}, +, \leqslant, 0)$ *generates the variety of lattice-ordered Abelian groups. That is, an equation in the language of lattice-ordered groups holds over the integers if and only if it holds in all lattice-ordered Abelian groups.*

We showed that this is an easy corollary of:

Lemma (Elliott's lemma [12]). *Let O be a totally ordered Abelian group, and let $P \subseteq O$ be a finite subset of strictly positive elements that generate O. Then there exists a finite subset $P' \subseteq O$ of strictly positive elements such that* (i) *P lies in the submonoid of O generated by P', and* (ii) *P' is linearly independent* (*in the $\mathbb{Z}$-module G*).

We sketched an elementary proof of Elliott's lemma, discussing how the subset P' whose existence is stated in the lemma can be constructed starting from the given P by taking successive subtractions of appropriate pairs of elements.[2]

Finally, we mentioned the MV-algebraic counterpart of Weinberg's Theorem.

Theorem (Chang's Completeness Theorem [9]). *The variety of MV-algebras is generated by the MV-algebra* $([0,1] \cap \mathbb{Q}, \oplus, \neg, 0)$*, where* $x \oplus y = \max\{x + y, 1\}$*, and* $\neg x = 1 - x$*, for each* $x, y \in [0,1] \cap \mathbb{Q}$*. That is, an equation in the language of MV-algebras holds in this MV-algebra if and only if it holds in all MV-algebras.*

Remark. Via Mundici's categorical equivalence (see Day 1), Chang's Completeness Theorem may be translated into an equivalent statement about unital Abelian ℓ-groups. In other words, Weinberg's Theorem does have a unital counterpart. Here it is: *an equation in the language of Abelian ℓ groups with a distinguished unit holds in all unital Abelian ℓ-groups if and only if it holds in* $(\mathbb{Q}, 1)$. It is not hard to see that this is equivalent to: *an equation in the language of Abelian ℓ-groups with a distinguished unit holds in all unital Abelian ℓ-groups if and only if it holds in* $(\mathbb{Z}, n)$*, for all* $n = 1, 2, \dots$. Comparing Weinberg's Theorem with the latter statement, you may gather that the presence of a unit does make a big difference when one is dealing with equational issues. That is indeed the case, as was discussed in depth during Charles Holland's mini-course.

[2] This proof of Weinberg's Theorem via Elliott's Lemma is given with full details in [23].

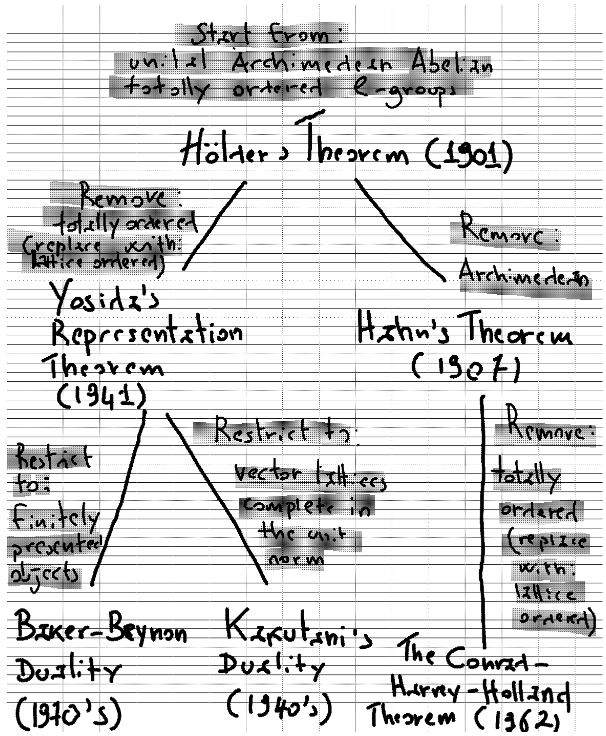

FIGURE 1. Hölder's Theorem and its descendants.

DAY 3 – WEDNESDAY, 10.VI.2009. In this lecture we discussed representations of Archimedean ℓ-groups by continuous real-valued functions. An ℓ-group G (not necessarily Abelian, but written additively) is *Archimedean* if whenever $0 \leqslant ng \leqslant h$ for two elements $g, h \in G$ and all integers $n \geqslant 1$, then $g = 0$. There can be no non-commutative examples of Archimedean ℓ-groups, hence the foregoing additive notation:

Theorem (Bernau's Theorem [3]). *Any Archimedean ℓ-group is Abelian.*

Rather than discussing the (non-trivial) proof of this theorem, we assumed throughout that all groups are Abelian. Additionally, we always assumed the existence of a unit, so as to simplify the representation theory. Among other things, Warren McGovern's mini-course dealt with the more general case of *weak order units*, where a satisfactory but more complex representation theory also is available.

To get started with the representation theory, we proved by elementary means:

Lemma (Hölder's Theorem for unital ℓ-groups [17][3]). *For any unital Archimedean totally ordered (necessarily Abelian) group (O, u) that is non-trivial (i.e. $O \neq \{u\}$), there exists a unique injective unital ℓ-homomorphism $h_O \colon (O, u) \rightarrowtail (\mathbb{R}, 1)$.*

Hölder's theorem may be extended in several directions, as indicated in Figure 1. In particular, we looked at the Yosida representation. For this, we associated with any unital Abelian ℓ-group (G, u) a topological space, as follows. Consider the set

[3]English translation in [18, 19].

of maximal ℓ-ideals $\operatorname{Max} G$ of G, and endow it with the *hull-kernel topology* having as a base of closed sets precisely those of the form

$$Z_I = \{\mathfrak{m} \in \operatorname{Max} G \mid \mathfrak{m} \supseteq I\} ,$$

as I ranges over all ℓ-ideals of G. Then $\operatorname{Max} G$ becomes a compact Hausdorff space. The topological space $\operatorname{Max} G$ is called the *maximal spectrum* of (G, u), after functional-analytic terminology; see [6] for a treatment of the spectrum. Finally, let us recall that a subset of functions $S \subseteq \mathrm{C}(X)$, for X a topological space, is called *separating* if for $x \neq y \in X$ there is $f \in S$ such that $f(x) \neq f(y)$.

Theorem (The Yosida representation [28])**.** *For any non-trivial unital Archimedean* (*necessarily Abelian*) *ℓ-group (G, u), the correspondence*

$$g \in G \longmapsto Y_G(g) ,$$

where $Y_G(g) \colon \operatorname{Max} G \to \mathbb{R}$ is the function given by

$$\mathfrak{m} \longmapsto h_{G/\mathfrak{m}}(g/\mathfrak{m}) \in \mathbb{R}$$

(*with $h_{G/\mathfrak{m}}$ as in Hölder's Theorem above*)*, yields an injective unital ℓ-homomorphism*

$$Y_G \colon (G, u) \rightarrowtail (\mathrm{C}(\operatorname{Max} G), 1)$$

onto a separating unital ℓ-subgroup of $(\mathrm{C}(\operatorname{Max} G), 1)$.

Remark. The representing space $\operatorname{Max} G$ is canonical, in the following sense. Suppose $E \colon (G, u) \rightarrowtail (\mathrm{C}(X), 1)$ is an injective unital ℓ-homomorphism onto a separating unital ℓ-subgroup of $(\mathrm{C}(X), 1)$, for some compact Hausdorff space X. Then X and $\operatorname{Max} G$ are homeomorphic spaces.

Together with further developments that we did not pursue, such as functoriality, this very important theorem shows that the nature of unital lattice-ordered *Abelian* groups essentially is that of algebras of continuous functions over a topological space. (The restriction to Archimedean objects can be lifted, in ways we did not discuss.) This should be contrasted with the case of general lattice-ordered groups, where the situation is radically different – a general lattice-ordered group essentially is a group of order-preserving permutations of an ordered set, as the Cayley-Holland Theorem shows; compare again Charles Holland's mini-course.

Day 4 – Thursday, 11.VI.2009. *Representations of free finitely generated unital Abelian ℓ-groups by piecewise linear functions.* In this lecture we started to move on from the general Yosida representation to more specialised versions for classes of unital Abelian ℓ-groups. First of all, we looked at free objects. A good place to start learning about free constructions at a fairly general – though still equational – level is [8]. As mentioned earlier, there are standard references such as [21] for the all-important abstraction of these universal-algebraic ideas at the level of general categories; and there is the less obvious source [22] that I warmly recommend.

Since MV-algebras form a variety, the free MV-algebra over any given generating set exists and is unique up to an isomorphism, by universal algebra. On the other hand, unital Abelian ℓ-groups are not definable by equations, so we cannot appeal to universal algebra to establish existence of free objects. We can, of course, translate the universal property of free MV-algebras into the analogous one for unital Abelian ℓ-groups using Mundici's categorical equivalence, and call the objects enjoying such a property 'free'. Here is the result.

Definition (Universal property of free objects)**.** Let S be any set. A unital Abelian ℓ-group (G, u) is *free over* S, or *freely generated by* S, if there is a function $\iota\colon S \to (H, u)$ with range contained in the unit interval $[0, u]$ such that whenever $\hat{f}\colon S \to (H, w)$ is any function with range contained in the unit interval $[0, w]$ of some unital Abelian ℓ-group (H, w), there exists a unique unital ℓ-homomorphism $f\colon (G, u) \to (H, w)$ such that

$$\hat{f} = f \circ \iota \ .$$

A free unital Abelian ℓ-group (G, u) over any given set S exists, and is in fact unique up to a unital ℓ-isomorphism. (This is just a translation of the analogous fact about MV-algebras, which, as mentioned, is granted by universal algebra.) Let us denote it (F_S, u), or (F_n, u) when S has finite cardinality $n \geqslant 0$. It turns out that the map ι enjoying the universal property in the definition is necessarily an injection, and that $\iota(S)$ necessarily is a generating set for the unital Abelian ℓ-group (F_S, u), hence the terminology in the definition. It follows that there is no harm in identifying ι with the inclusion map, and we can therefore think of $S \subseteq [0, u] \subseteq F_S$ as a special generating set for (F_S, u). (Here, it should be noted that to say that $S \subseteq [0, u] \subseteq F_S$ generates (F_S, u) means that $S \cup \{u\}$ generates F_S as a lattice-ordered group.) Then the content of the definition may be equivalently rephrased in a way that might ring more familiar to you – say, from group theory.

Definition (Free objects, rephrased)**.** Let (G, u) be a unital Abelian ℓ-group, and let $S \subseteq [0, u]$ be a subset. Then (G, u) is *free over* S if (i) $S \cup \{u\}$ generates the ℓ-group G, and (ii) whenever (H, w) is a unital Abelian ℓ-group and $\hat{f}\colon S \to [0, w]$ is any function, there exists a unique ℓ-homomorphism $f\colon (G, u) \to (H, w)$ that agrees with $\hat{f}$ on S.

Remark. The definition above of free unital Abelian ℓ-group is devised so that $\Gamma(F_S, u)$ is the free MV-algebra over S, which explains why the range of ι must be taken to be $[0, u]$ and not the whole of F_S. However, one should bear in mind that *F_S is not a free Abelian ℓ-group in the variety of all Abelian ℓ-groups* (without a distinguished unit). In fact, it can be shown that $\operatorname{Max} F_n$ is homeomorphic to an n-dimensional cube, whereas if A_n denotes the free Abelian ℓ-group on n generators, $\operatorname{Max} A_n$ is homeomorphic to an $(n-1)$-dimensional sphere. We did not discuss this point any further for lack of time.

After these preliminaries, we discussed a representation theorem for free unital Abelian ℓ-groups. For this, we introduced piecewise linear functions.[4] Recall that a function $f\colon [0, 1]^n \to \mathbb{R}$ is (*finitely*) *piecewise linear* if it is continuous, and there exists a finite set of (affine) linear functions $l_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, such that for each $x \in [0, 1]^n$ one has $f(x) = l_i(x)$ for some $1 \leqslant i \leqslant k$. Here, $[0, 1]^n$ is the real unit n-cube with its usual Euclidean topology. Let us say such a piecewise linear function $f\colon [0, 1]^n \to \mathbb{R}$ has *integer coefficients* if in the preceding definition one can take each l_i to have integer coefficients, i.e. to be expressible as a linear form $l_i(x_1, \ldots, x_n) = z_0 + z_1 x_1 + \cdots + z_n x_n$ with $z_j \in \mathbb{Z}$. More generally, if $X \subseteq [0, 1]^n$ is a subset of the real unit n-cube with the subspace topology, we say a function $f\colon X \to \mathbb{R}$ is *piecewise linear with integer coefficients* if it is the restriction to X of some such function defined over the whole of $[0, 1]^n$. Let us write $\nabla(X) \subseteq \mathrm{C}(X)$

[4] Piecewise linear functions are investigated in piecewise linear topology and geometry, a subject with an extensive literature; one standard reference is [26].

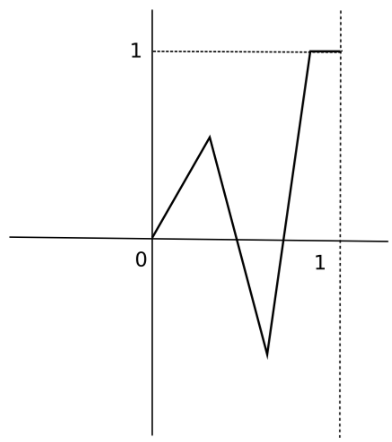

FIGURE 2. A piecewise linear function $f\colon [0,1] \to \mathbb{R}$.

for the unital ℓ-subgroup of $\mathrm{C}(X)$ consisting of all piecewise linear functions with integer coefficients. It may be proved that $\nabla(X)$ indeed is a sublattice and a subgroup of $\mathrm{C}(X)$, and it is clear that the unit 1 of $\mathrm{C}(X)$ (the function constantly equal to 1) does belong to $\nabla(X)$.

Theorem (The unital version of Baker's and Beynon's representation of finitely generated free Abelian ℓ-groups [2, 4, 5])**.** *Let (F_n, u) be the unital Abelian ℓ-group freely generated by $S = \{g_1, \ldots, g_n\} \subseteq [0, u]$, for $n \geqslant 1$ an integer. Let $\pi_i\colon [0,1]^n \to \mathbb{R}$, $i = 1, \ldots, n$, be the i^{th} projection function $\pi(x_1, \ldots, x_n) = x_i$. The correspondence*

$$g_i \mapsto \pi_i$$

has a unique extension to a unital ℓ-isomorphism

$$Y_S\colon (F_n, u) \to \nabla([0,1]^n) \ .$$

That one such extension Y_S exists is guaranteed by the universal property of (F_n, u); by the same token, we know the extension must be unique. The content of the theorem is that Y_S actually is a *bijection* onto $\nabla([0,1]^n)$. One half of this – injectivity – is readily reduced using standard universal algebra to the unital version of Weinberg's Theorem, that is, to Chang's Completeness Theorem; for the non-unital version in the special case of vector lattices, see Baker's paper. The other half – surjectivity – is due, in the non-unital case, to Beynon. The MV-algebraic counterpart of Beynon's result is known as *McNaughton's Theorem* [24]. Finally, I am using the notation Y_S here because, as discussed during the lecture, Y_S is essentially the same thing as the Yosida representation Y_{F_n}, and in fact one can use Yosida's result to obtain a proof[5] of the Baker-Beynon representation. The main differences are

[5]However, to apply to (F_n, u) Yosida's result we need to know that F_n is Archimedean. This, again, is a simple consequence of Chang's Completeness Theorem, modulo standard universal algebra. Here let me just mention that Weinberg's Theorem may be rephrased as saying that the free Abelian ℓ-group over n generators, $n \geqslant 0$ an integer, is a subdirect product of copies of $\mathbb{Z}$,

that here Y_S is constructed using the given free generating set S for F_n to induce the embedding $F_n \to \mathrm{C}([0,1]^n)$, while in the general case the Yosida embedding Y_{F_n} is constructed using all of the elements of F_n as a (very redundant) generating set; and, moreover, here we are identifying the exact range of Y_{F_n} inside $\mathrm{C}([0,1]^n)$, whereas this cannot be done in the general setting of Yosida's result.

Day 5 – Friday, 12.VI.2009. *Baker-Beynon Duality for finitely presented unital Abelian ℓ-groups.* In the light of the preceding lecture, in this last lecture we identified (F_n, u) with $\nabla([0,1]^n)$; let us just write ∇_n to denote the latter unital Abelian ℓ-group, for short.

By the remark following the Yosida representation above, along with the easily established fact that ∇_n is a separating ℓ-subgroup of $\mathrm{C}([0,1]^n)$, we know that $[0,1]^n$ with its standard Euclidean topology is homeomorphic to $\mathrm{Max}\,\nabla_n$. In particular, this means that the maximal ℓ-ideals of ∇_n are in bijection with the points of $[0,1]^n$; what about the remaining ℓ-ideals? For each subset $I \subseteq \nabla_n$ (not necessarily an ℓ-ideal), consider the *vanishing locus*

$$\mathbb{V}(I) = \{x \in [0,1]^n \mid f(x) = 0 \text{ for all } f \in \nabla_n\} \ .$$

One checks (using continuity) that $\mathbb{V}(I)$ is a closed subset of $[0,1]^n$; further, $\mathbb{V}(I) = \mathbb{V}(\langle I \rangle)$ if $\langle I \rangle$ denotes the ℓ-ideal generated[6] by I. Thus, *we associate a closed subset of $[0,1]^n$ with each ℓ-ideal of ∇_n.*

Conversely, for a (not necessarily closed) subset $K \subseteq [0,1]^n$, consider the *ℓ-ideal of elements vanishing over K*, namely

$$\mathbb{I}(K) = \{f \in \nabla_n \mid f(x) = 0 \text{ for all } x \in K\} \ .$$

One checks that $\mathbb{I}(K)$ actually is an ℓ-ideal of ∇_n; further, $\mathbb{I}(K) = \mathbb{V}(\overline{K})$ if $\overline{K}$ denotes the closure of K. Thus, *we associate an ℓ-ideal of ∇_n with each closed subset of $[0,1]^n$.*

Let us write $\mathrm{Con}\,\nabla_n$ for the collection of all ℓ-ideals of ∇_n, and $\mathscr{K}_n$ for the collection of all closed subsets of $[0,1]^n$. These collections are in fact complete lattices under inclusion. The operators

$$\mathbb{I}\colon \mathscr{K}_n \longrightarrow \mathrm{Con}\,\nabla_n \quad , \quad \mathbb{V}\colon \mathrm{Con}\,\nabla_n \longrightarrow \mathscr{K}_n$$

are order-reversing, and yield a *Galois connection* between the two lattices at hand. We did not recall nor use the general theory of Galois connections, but we did observe the following. If we compose $\mathbb{I}$ (applied first) with $\mathbb{V}$ (applied second), we obtain the identity. That is, *for any closed set $K \subseteq [0,1]^n$, we have*

$$\mathbb{V}(\mathbb{I}(K)) = K \ .$$

On the other hand, *if we start with an ℓ-ideal I, then we only get*

$$\mathbb{I}(\mathbb{V}(I)) \supseteq I \ ,$$

and we gave examples to show that the inclusion may well be proper. (Just choose for I the prime ℓ-ideal "to the right of 0" in ∇_1, i.e. I consists of those functions that vanish in an open neighbourhood of 0; then $\mathbb{I}(\mathbb{V}(I))$ is the maximal ℓ-ideal of functions vanishing at 0, which is strictly larger than I.) *The ideals I such*

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hence obviously Archimedean. Similar considerations apply in the unital case to (F_n, u). We thus see once more that Weinberg's (or Chang's) Theorem really is a fundamental fact.

[6]Generated *as an ℓ-ideal*, and not merely as a sublattice subgroup: by definition, $S \subseteq G$ generates the ℓ-ideal given by the intersection of all ℓ-ideals of G containing S.

that $\mathbb{I}(\mathbb{V}(I)) = I$ *are precisely those that may be represented as an intersection of maximal* ℓ*-ideals in* $\mathrm{Con}\,\nabla_n$*.* Equivalently, $\mathbb{I}(\mathbb{V}(I)) = I$ *if and only if* ∇_n/I *is Archimedean.*

Suppose now $I \in \mathrm{Con}\,\nabla_n$ happens to be an intersection of maximal ℓ-ideals of ∇_n. Then ∇_n/I is unital Archimedean, and thus has a Yosida representation in terms of continuous functions on its maximal spectrum. How does this latter representation relate to the one for F_n, i.e. to ∇_n? Very easily. *Assume* $I \in \mathrm{Con}\,\nabla_n$ *is an intersection of maximal* ℓ*-ideals of* ∇_n*. Then* ∇_n/I *is* ℓ*-isomorphic to* $\nabla(\mathbb{V}(I))$*, and the natural quotient map*

$$\nabla_n \twoheadrightarrow \nabla_n/I$$

corresponds to the restriction map

$$f \longmapsto f\restriction\mathbb{V}(I)\ .$$

In words: to visualise a general Archimedean quotient of ∇_n, just select a closed set $K \subseteq [0,1]^n$, and restrict all functions in ∇_n to K. (Non-Archimedean quotients can also be understood in terms of this functional representation, although we did not cover this in any detail.)

Principal ℓ*-ideals*, that is, ℓ-ideals generated by a single element, are of special importance to Baker-Beynon duality. An ℓ-ideal I of an ℓ-group G is principal if and only if there is $g \in G^+$ lying in I such that $I^+ = \{x \in G \mid 0 \leqslant x \leqslant ng$ for some integer $n \geqslant 0\}$. Observe that this can be rephrased as follows: I is principal if and only if, as an ℓ-group in its own right, I has a unit – namely, g. We remarked that finitely generated ℓ-ideals are the same thing as principal ℓ-ideals – something that fails, e.g., for normal subroups of groups. A unital Abelian ℓ-group is *finitely presented* if it is ℓ-isomorphic to a quotient F_n/I, for I a principal ℓ-ideal. As we briefly explained, this terminology comes from the fact that such ℓ-groups can be presented by a finite set of generators along with a finite set of relations that they satisfy in the language of unital ℓ-groups. A non-trivial argument shows that *every finitely presented unital Abelian* ℓ*-group is Archimedean.*

Finitely presented unital Abelian ℓ-groups are visualised as follows in terms of ∇_n. Let I be a principal ℓ-ideal of ∇_n, and let g be a unit of I. Then $\mathbb{V}(I) = g^{-1}(0)$. That is, the closed set corresponding to I is just the *zero set* of g, i.e. the locus of points in $[0,1]^n$ where g vanishes. But since g is piecewise linear with integer coefficients, it follows that $g^{-1}(0)$ is a rational polyhedron in $[0,1]^n$; conversely, any rational polyhedron in $[0,1]^n$ is the zero set of some $g \in \nabla_n$.

To explain what this means we recall some basic definitions from convex geometry. A *convex combination* of vectors $v_1,\ldots,v_m \in \mathbb{R}^n$ is an element $x \in \mathbb{R}^n$ of the form

$$x = \lambda_1 v_1 + \cdots + \lambda_m v_m\ , \quad \text{for } \lambda_i \in \mathbb{R} \text{ with } \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1$$

(Geometrically, the convex combinations of two vectors $v_1, v_2 \in \mathbb{R}^n$ are precisely the points of the segment joining v_1 and v_2; as extreme cases, v_i corresponds to $\lambda_i = 1$ and $\lambda_j = 0$ for $j \neq i$. For three vectors v_1, v_2, v_3 not on a line, the convex combinations yield the triangle whose vertices are precisely v_1, v_2, v_3; and so on.) The *convex hull* of $v_1,\ldots,v_m$ is the set of all their convex combinations. A *polytope* P in $\mathbb{R}^n$ is the convex hull of finitely many vectors; it is *rational* if it may be written as the convex hull of finitely many vectors whose coordinates are rational numbers. Finally, a (*rational*) *polyhedron*[7] is the union of finitely many (rational) polytopes.

Returning now to the representation of finitely presented unital Abelian ℓ-groups in terms of ∇_n, we have the following important fact. *A unital Abelian ℓ-group (G, u) is finitely presented if and only if there exists a rational polyhedron $P \subseteq [0,1]^n$, for some choice of the integer $n \geqslant 1$, such that there is a unital ℓ-isomorphism*

$$(G, u) \cong \nabla(P) \ . \tag{1}$$

Let me repeat how this statement relates to our previous discussion of $\mathbb{I}$ and $\mathbb{V}$. Let us start from a polyhedron $P \subseteq [0,1]^n$. Then P is a (very special) closed set, and one proves that $\mathbb{I}(P)$ not only is an intersection of maximal ℓ-ideals, but is in fact a principal ℓ-ideal. The quotient $\nabla_n/\mathbb{I}(P) \cong \nabla(P)$ is therefore a unital finitely presented Abelian ℓ-group. Conversely, if we start from a finitely presented object (G, u), there is a principal ℓ-ideal $I = \langle g \rangle$ such that $\nabla_n/\langle g \rangle \cong (G, u)$. Since G is Archimedean, i.e. $\langle g \rangle$ is an intersection of maximal ideals, we have $\nabla_n/\langle g \rangle \cong \nabla(\mathbb{V}(\langle g \rangle)) \cong \nabla(g^{-1}(0))$, where $g^{-1}(0) = P$ is a rational polyhedron in $[0,1]^n$.

There is one last step needed to upgrade the representation theorem for finitely presented objects just discussed to Baker-Beynon duality. Consider two finitely presented unital Abelian ℓ-groups (G, u) and (H, w). We know that there are rational polyhedra P and Q representing them as in (1), respectively. However, suppose we are also given a unital ℓ-homomorphism $f\colon (G, u) \to (H, w)$; is there a geometric counterpart of f in terms of some transformation between P and Q (or Q and P)? Baker-Beynon duality completely answers this question in the affirmative. (Note that if the answer were negative, the representation theorem in terms of polyhedra discussed above would have modest import. To appreciate this point, take f to be an ℓ-isomorphism. Then P and Q may well be chosen to be different polyhedra, and the representation given by (1) would surely be of little value if we could not relate P and Q at all. In fact, in this case we would like P and Q to be "essentially the same" in *some* appropriate sense, so as to reflect the fact that (G, u) and (H, w) indeed are the same to within the notion of isomorphism that is natural for ℓ-groups.)

Let $P \subseteq [0,1]^m$ and $Q \subseteq [0,1]^n$ be polyhedra, and let $F\colon P \to Q$ be a continuous map. Then there exist uniquely determined continuous functions

$$f_1, \ldots, f_n\colon P \to [0,1]$$

such that for every $(x_1, \ldots, x_m) \in P$ we have

$$F(x_1, \ldots, x_m) = (\, f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\,) \ .$$

Explicitly, f_i is obtained composing F with the projection onto the i^{th} coordinate $\pi_i\colon [0,1]^n \to [0,1]$, $i = 1, \ldots, n$. If each f_i may be taken to be piecewise linear with integer coefficients, F is said to be a *piecewise linear map with integer coefficients*. Piecewise linear maps with integer coefficients between rational polyhedra may be composed, and the identity always is such a map; hence, we may consider the category $\mathsf{RatPoly}_{\mathbb{Z}}$ whose objects are rational polyhedra (each lying in some $[0,1]^n$), and whose morphisms are piecewise linear maps with integer coefficients

[7]This is the standard terminology in piecewise linear geometry. In other contexts, though, usage may differ. In classical geometry, for instance, a polyhedron is just what we are calling a polytope.

between such polyhedra. Also, let us write $\mathsf{UnAbLatGp}_{\mathrm{fp}}$ for the category of finitely presented unital Abelian ℓ-groups and their unital ℓ-homomorphisms. Finally, for any category C, let us write C^{op} for its *opposite category*: the objects of C^{op} are just the same as the objects of C; and C^{op} has exactly one morphism $A \to B$ for each morphism $B \to A$ that C has. Informally, C^{op} is obtained from C by "reversing the arrows". If D and C^{op} are equivalent categories, then one says that D and C are *dually equivalent*, or that there is a *duality* between them. Here is the final result relating polyhedra and ℓ-groups, whose proof is based on all previous results in this lecture:

Theorem (Baker-Beynon Duality for finitely presented unital Abelian ℓ-groups)**.** *The categories* $\mathsf{AbUnLatGp}_{\mathrm{fp}}$ *and* $\mathsf{RatPoly}_{\mathbb{Z}}$ *are dually equivalent.*

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