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On the local action of automorphisms of free finitely generated MV-algebras on maximal ideals

Stefano Aguzzoli*; Brunella Gerla; Vincenzo Marra

Abstract

Fix an integer $n \geq 1$, and write F_n for the MV-algebra (= unit interval of unital Abelian ℓ-group) that is free over n generators. Any choice of a free generating set of F_n induces a unique homeomorphism between the space of maximal ideals of F_n and the real unit n-cube $[0,1]^n$ with the Euclidean topology. Further, (the opposite of) the automorphism group $Aut(F_n)$ of F_n acts faithfully on $[0,1]^n$ by self-homeomorphisms. This action is far from being transitive. For one thing, there are obvious topological constraints – any self-homeomorphism of $[0,1]^n$ will leave its boundary invariant. However, even within the topological interior of $[0,1]^n$, there are subtler, non-topological obstructions to transitivity. Let us call a maximal ideal $\mathfrak{m} \subseteq F_n$ *rational* if $F_n/\mathfrak{m}$ has finite cardinality, say $d+1$; in this case, d is the *denominator* of $\mathfrak{m}$. This terminology is justified by the fact that whenever we identify $[0,1]^n$ with the space of maximal ideals of F_n as in the above, $\mathfrak{m}$ is rational of denominator d if and only if $\mathfrak{m} \in [0,1] \cap \mathbb{Q}^n$ and d is the l.c.m. of the denominators of the rational coordinates of $\mathfrak{m}$ (written in reduced form). One checks that the action of any element of $Aut(F_n)$ must carry a maximal ideal $\mathfrak{m}$ of denominator d to a maximal ideal with the same denominator. The converse is a nontrivial open problem: it is not known whether $Aut(F_n)$ acts transitively on the set of rational points in $(0,1)^n$ having the same denominator.

Using available knowledge about the prime, maximal, and germinal ideals of free finitely generated lattice-ordered Abelian groups ([2], [1]), we introduce an appropriate local version of the preceding question that might provide a more tractable case, and is interesting in its own right. Beyond the case $n = 1$, where everything is understood, even in the local case a complete solution is not available at the time of writing; however, we do obtain some encouraging partial results.

References

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The Conrad-Harvey-Holland Theorem

Rick Ball

Abstract

Paul Conrad's first two PhD students were John Harvey and Charles Holland, who received their degrees simultaneously. (Both served as my PhD advisors, titular in the case of the former and de facto in the case of the latter.) Conrad's graduation gift was to propose the problem of representing abelian lattice ordered groups in the spirit of Hahn's famous representation of abelian totally ordered groups. The resulting theorem forms the basis for all we really know about the structure of general abelian lattice ordered groups. In this talk I will discuss this theorem in the light of Priestley duality.

Cold-filtered ℓ-groups

Tumpa Bhattacharyya

Abstract

Suppose $(G, +, 0)$ is an ℓ-group and $\mathfrak{F}$ is a filter on G^+. Recall that $\mathfrak{F}$ is a *principal filter* if it is of the form $\{g \in G^+ : a \leq g\}$ for some $a \in G^+$. $\mathfrak{F}$ is said to be a *cold filter* if for all $P \in Min(G)$, the filter (i.e. interval) on G/P^+ defined by $\mathfrak{F}_P = \{g + P : g \in \mathfrak{F}\}$ has a minimum. If every filter on G^+ is a principal (resp. cold) filter then the group is called a *principally-filtered (resp. cold-filtered)* ℓ-group. A principally-filtered ℓ-group is isomorphic to direct sum of copies of $\mathbb{Z}$. We shall discuss the characterizations of cold-filtered ℓ-groups.

An extension of the duality between locally finite MV-algebras and multisets

Roberto Cignoli

Abstract

In [1] a duality was established between locally finite MV-algebras and multisets. A multiset was defined as a continuous mapping from a Boolean space into the lattice of generalized natural numbers equipped with the Scott topology. This lattice is isomorphic to the lattice of the unitary subgroups of $\mathbb{Q}$, the additive group of the rational numbers endowed with the usual order [2]. A natural problem is to investigate the generalization of the above duality when Q is replaced by R, the additive group of the real numbers equipped with the usual order. The aim of the talk is to present results obtained in collaboration with Enzo Marra about this generalized duality.

References

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[2] E. G. Effros, Dimensions and C^*-algebras, American Mathematical Society, Providence, RI., 1980.

Characterizing Classes of Topological Spaces by Frame Identities Involving Pseudocomplement and Heyting Implication

Fred Dashiell

Abstract

An example of the type of frame identity discussed is:

$$(a \vee b)^{**} = a^{**} \vee b^{**} \tag{1}$$

In a bounded, pseudocomplemented, distributive lattice, it is well-known that this identity characterizes Stone lattices. In a completely regular frame L, we show that requiring (1) to hold only in the cozero part Coz(L), instead of in all of L, characterizes the quasi-F frames. Similarly, if (1) is required to hold for arbitrary $a \in$ L and only $b \in$ Coz(L); you get a characterization of basically disconnected frames. A second example is the identity

$$(a \to b) \vee (b \to a) = 1 \tag{2}$$

The identity (2) is known to characterize relative Stone lattices. In a completely regular frame L; by restricting (2) to Coz(L) you get a characterization of F'-frames, and in addition requiring the join in (2) to have a cozero refinement which also covers 1, you get a characterization of F-frames. If you further restrict (2) to hold only for $a, b \in$ Coz(L) with $a \vee b$ dense, you get an additional characterization of quasi-F frames, which in this case is just equivalent to requiring the binary cover to have a cozero refinement. If you require (2) to hold for $a \in$ Coz(L) and $a \vee b$ dense (with b arbitrary), you get a characterization of basically disconnected frames. Denoting the dense element $x \vee x^*$ by $\hat{x}$; the identity

$$(a \to b) \vee (b \to a) = (\hat{a} \to \hat{b}) \vee (\hat{b} \vee \hat{a}) \tag{3}$$

characterizes the Stone lattices among Heyting algebras. (This appears to be new, even for spaces.) We also discuss similar considerations involving the de Morgan identity

$$(a \wedge b)^* = a^* \vee b^*.$$

We discuss the preservation of these frame properties under quotient frame maps or dense quotient frame maps.

Automorphism groups of totally ordered sets: a retrospective survey

Manfred Droste*; A.M.W. Glass

Abstract

In 1963, W. Charles Holland proved that every lattice-ordered group can be embedded in the lattice-ordered group of all order-preserving permutations of a totally ordered set. We examine the context and proof of this result and survey some of the many consequences of the ideas involved in this important theorem.

An N2-variety of residuated lattices that is not closed under completions.

Nikolaos Galatos*; A. Ciabattoni; K. Terui

Abstract

Residuated lattices are common generalizations of ℓ-groups, MV-algebras and Heyting algebras, and their equational theory is known to be decidable. An open research project asks about sufficient conditions for other subvarieties to have a decidable equational theory, as well. A syntactic approach to the problem, connected to substructural logics, involves Gentzen-style sequent deductive systems and attempts to rewrite a given equation as an equivalent structural rule of the deductive system. A necessary step in this approach to obtaining decidability is that the rule is analytic. Analytic rules are important for various other reasons, as well.

Equations are classified according to a hierarchy and sequent structural rules correspond to the $\mathcal{N}_2$ level of the hierarchy. A fair amount of $\mathcal{N}_2$ equations, the ones that satisfy the syntactic criterion of acyclicity, correspond directly to structural rules that are analytic. Other $\mathcal{N}_2$ equations, even though not acyclic themselves, are equivalent to acyclic equations, and thus also yield analytic structural rules. Among $\mathcal{N}_2$ equations, the ones equivalent to acyclic equations are exactly the ones that are preserved under (Dedekind-MacNeille/any) completions.

Even though many $\mathcal{N}_2$ equations are expected to fail to be equivalent to an acyclic one, the problem of proving that one exists was particularly resilient to resolution. I will present one example of such an equation, the only one we know so far. The proof requires a combination of non-trivial proof-theoretic and algebraic results. The algebraic side considers the totally ordered ℓ-group based on the free group on two generators and some of its properties established by G. Bergman in 1984.

Local MV-algebras with finite rank and unital ℓ-groups

Brunella Gerla

Abstract

Local MV-algebras, i.e., MV-algebras having only one maximal ideal, have been investigated in [1] and later on in [4]. In [3] local MV-algebras with finite ranks have been characterized as those MV-algebras corresponding, via the Mundici's Γ functor, to the unital ℓ-groups $(\mathbb{Z}\overrightarrow{\times}G, (n, g))$, where G is any abelian ℓ-group, $\overrightarrow{\times}$ is the lexicographic product and $0 < g \in G$. A generalization to the non-commutative case has been investigated in [5]. We recall that an MV-algebra A has finite rank n if and only if $A/Rad(A)$ is isomorphic to the linearly ordered MV-algebra $\text{Ł}_n = \{0, 1/n, \ldots, (n-1)/n, 1\}$, where $Rad(A)$ is the intersection of maximal ideals of A.

A prototypical example of local MV-algebra is the algebra of quasi-constant functions: let X be an arbitrary nonempty set, U an MV-algebra, and $\mathbf{K}(U^X)$ the subset of the MV-algebra U^X defined as follows:

$$\mathbf{K}(U^X) = \{f \in U^X \mid f(X) \subseteq a/Rad(U) \quad \text{for some} \quad a \in U\}.$$

Any element f of $\mathbf{K}(U^X)$ will be called a *quasi-constant* function from X to U.

The MV-algebra $\mathbf{K}(U^X)$ is local and every local MV-algebra can be embedded into an MV-algebra of *quasi-constant* functions ([4]). Further, if A is a local MV-algebra with rank n, then A can be embedded in $\mathbf{K}(U^X)$ where $U/Rad(U) \cong \text{Ł}_n$.

Generalizing the work begun in [2], we shall describe the class of unital ℓ-groups corresponding to local MV-algebras with finite rank, exploiting the characterization in terms of quasi-constant functions.

References

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Quasivarieties of MV-algebras

Joan Gispert

Abstract

In this talk we will review the characterization and classification of some quasivarieties of MV-algebras. In [1], Komori gives a classification and characterization of all subvarieties of the class all MV-algebras. He proves that a variety generated by a single MV-chain is uniquely characterized by its order and its rank. The purpose of this talk is to give a similar characterization and classification for some quasivarieties of MV-algebras. We will show that, for the case of quasivarieties generated by a single MV-chain, the set of all rationale elements of the algebra turns out to be the third invariant together with the order and the rank of the algebra. We will study the relation among rationale elements and MV-algebras, raise some questions on quasiequational definability and solve some of them.

References

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On hulls of archimedean ℓ-groups with unit

Anthony Hager

Abstract

$\mathbf{W}$ is the category of archimedean ℓ-groups with distinguished weak order unit. Each $\mathbf{W}$-object G has its compact Yosida space YG (of values of the unit), and its Yosida representation in $D(YG)$ (the extended-real functions). Let $\mathfrak{F}$ be a filter base of dense sets in YG which contains all reality sets of the elements of G, and $C[\mathfrak{F}]$ the direct limit in $\mathbf{W}$ of $C(S)$ over $S \in \mathfrak{F}$. Then, $C[F]$ is an essential extension of G, and the maximum essential extension of G is obtained using $\mathfrak{G}$, the collection of all countable intersections of dense open sets. In fact, for many of the canonical hulls of G, there is an $\mathfrak{F}$ for which the hull is either all of, or "very dense" in $C[\mathfrak{F}]$. We catalogue and systematize the situation.

Some remarks on P-spaces and the prime ideals of $C(X)$

Melvin Henriksen

Abstract

A brief survey of recent results will be given.

Top and bottom varieties of unital ℓ-groups and pseudo MV-algebras

W. Charles Holland

Abstract

Unital ℓ-groups and pseudo-MV-algebras are categorically equivalent, and so their varieties (equationally defined classes) correspond with each other. The varieties which are currently best understood are: those near the bottom, in the layer which covers the Boolean variety; and those near the top, which are generated by primitive unital ℓ-permutation groups of the real line. I will discuss what is known about both of these, and present open problems in both regions.

Computing in lattice ordered groups and related structures

Peter Jipsen

Abstract

Abstract: Most computer languages provide native implementations of a subset of the integers with $+, -, \min, \max$, as well as approximations of real numbers. Calculations in finitely generated free abelian ℓ-groups are possible with piecewise linear functions, and computing in non-abelian ℓ-groups requires additional effort to implement, making use of Charles Holland's embedding into ℓ-groups of order-automorphisms of a chain. We will present a JavaScript/Python implementation of the Holland-McCleary algorithm for deciding ℓ-group equations and we discuss the use of automated and interactive theorem provers such as Prover9 and Isabelle for proving results about ℓ-groups and related structures. From joint research with N. Galatos, we also give an embedding of ℓ-groups into distributive residuated lattices of binary relations with composition as multiplication. This construction generalizes order-automorphisms of a chain, and provides new examples of involutive residuated lattices that are "close" to ℓ-groups but are non-cyclic.

Strongly clean matrix rings over $C(X)$

Ramiro Lafuente*; Wolf Iberkleid; Warren Wm. McGovern

Abstract

We study some special cases of rings of $n \times n$-matrices over $C(X)$ to determine when they are clean and when they are strongly clean. An element of a ring is *clean* if it can be expressed as the sum of an idempotent and a unit. The element is *strongly clean* if it can be expressed as the sum of an idempotent and a unit which commute. A ring is called (strongly) clean if all of its elements are (strongly) clean.

Bases in lattice-ordered Abelian groups and MV-algebras: a survey

Vincenzo Marra

Abstract

Bases of lattice-ordered Abelian groups were introduced in the speakers 2002 Ph.D. thesis as an algebraic abstraction of (geometric) Schauder bases. They have since found a number of applications. The purpose of this talk is to survey these developments. We mainly concentrate on the case of unital lattice-ordered Abelian groups (= MV-algebras), although a parallel non-unital version of the theory is available. We first discuss several possible definitions of bases, each highlighting a different aspect of the notion. Proving that these definitions are all equivalent is a definitely non-trivial task; if time allows, we sketch some of the arguments involved. We then survey applications to (i) measure theory over MV-algebras, (ii) MV-algebras that are free over finite distributive lattices and Kleene algebras, and (iii) complete isomorphism invariants of finitely presented MV-algebras.

Torsion Classes in ℓ-Groups with Strong Unit

Jorge Martínez

Abstract

In a concrete category $\mathfrak{C}$ with image factorization, call a full subcategory $\mathfrak{T}$ a *torsion class* if it is closed under the formation of subobjects, coproducts, and images by surjective maps of $\mathfrak{C}$. We will discuss these torsion classes when the objects in them are "archimedean" and "compact", in the context of frames and ℓ-groups.

W. Charles Holland

Warren Wm. McGovern

Finitely presented and finitely generated projective MV-algebras and unital l-groups.

Daniele Mundici

Abstract

Despite the archimedean property of order units is undefinable in first-order logic, recent work of Holland and others has shown that unital ℓl-groups (i.e., lattice-ordered groups with a distinguished order unit) form a variety in a natural sense. In particular, his results on the classification of top subvarieties show that the order unit enriches any lattice-ordered group with a surprisingly complex structure. One has a similar experience by considering, in the opposite direction, certain small classes of finitely generated abelian unital ℓ-groups (i.e., MV-algebras). Thus for instance, almost nothing is known about the decidability of the isomorphism problem for finitely presented unital abelian ℓ-groups, or the characterization of projectives—notwithstanding Markov's unrecognizability theorem for finitely presented abelian ℓ-groups, and the Baker-Beynon theorem stating that an abelian ℓ-group is finitely generated projective iff it is finitely presented. I will present some new results on these two open problems.

Intrinsic generalized metrics on ℓ-groups

Homeira Pajoohesh

Abstract

Charles Holland showed that the intrinsic metrics from a lattice ordered group to itself are exactly the functions of the form

$$d(x, y) = n|x - y|$$

for some integer n, and that for $n \geq 1$, the triangle inequality for these functions occurs if and only if the group is abelian. In this talk we introduce intrinsic partial metrics and intrinsic quasimetrics and discuss similar representations for intrinsic partial metrics and quasimetrics on lattice ordered groups.

Galois strong units in free l-groups

Giovanni Panti

Abstract

A field extension K/F is Galois if the subfield fixed by $Aut(K/F)$ is F. Let us then define a strong unit u in the abelian ℓ-group G to be Galois if the only strong units fixed by $Aut(G,u)$ are u and its multiples. We will present some results and open problems on Galois strong units in free abelian ℓ-groups.

On orderable solvable groups in which all right orders are Conradian

Akbar Rhemtulla*; Vasily Bludov

Abstract

A right order $\leq$ on a group G is called Conradian if for every $1 \leq a$ and $1 \leq b$ there exists some integer $n > 0$ such that $a \leq a^n b$. We consider orderable groups in which every right order is Conradian. It is shown that a finitely generated solvable group G has this property if and only if G is torsion-free nilpotent. If the requirement that G be solvable is removed then G need not be nilpotent. In such cases G has a non-solvable residually torsion-free nilpotent quotient and every two sided order on G is central. Examples where this occurs are given. The case where G is not finitely generated is discussed and it is shown that a abelian-by-nilpotent orderable group in which every right order is Conradian is locally nilpotent. A number of open questions are listed.

Semirings and Semimodules in Many-Valued Logics

Ciro Russo

Abstract

A *semiring* [3] is a structure $\mathbf{S} = \langle S, +, \cdot, 0, 1 \rangle$ such that $\langle S, +, 0 \rangle$ is a commutative monoid, $\langle S, \cdot, 1 \rangle$ is a monoid and $\cdot$ distributes over $+$ from either side. A *semimodule* over a semiring is a commutative monoid subject to an action from the semiring that fulfils the usual module conditions.

In [1, 2] the authors showed a deep connection between MV-algebras and a special category of additively idempotent semirings. Indeed, on the one hand, every MV-algebra $\mathbf{A}$ has two *semiring reducts* — $\mathbf{A}^\vee = \langle A, \vee, \odot, 0, 1 \rangle$ and $\mathbf{A}^\wedge = \langle A, \wedge, \oplus, 1, 0 \rangle$ — that are isomorphic under the involution *; on the other hand, the category of *MV-semirings*, or *Lukasiewicz semirings*, defined in [1], is isomorphic to the one of MV-algebras. Moreover, such results naturally led to applications of MV-semirings and MV-semimodules to automata theory (see [4]); about these semimodules, it is worthwhile noticing that semimodules over additively idempotent semirings are idempotent themselves, thus they are actually lower bounded join-semilattices.

In this talk, we will show there is an extension of the Γ functor to MV-semimodules (i.e. semimodules over MV-semirings satisfying an additional condition) and a representation of MV-semirings – hence of MV-algebras – as subalgebras of the semiring of the endomorphisms of a join-semilattice.

Moreover, we will give a glimpse of possible applications of MV-semimodules to logic.

References

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Strongly adequate model of Łukasiewicz's logic via MV-algebra embeddings

Piotr J. Wojciechowski

Abstract

According to Chang's Completeness Theorem, the variety of all MV-algebras is generated by a single MV-algebra, the standard interval $[0, 1]$. We will show that as a *quasivariety*, the MV-algebras are also generated by a single structure and will provide its construction. We will use several equivalent conditions for a quasivariety to be generated by a single structure. For a given cardinal number $\mathfrak{a}$, we construct a totally-ordered MV-algebra $M(\mathfrak{a})$ having the property that every totally-ordered MV-algebra of cardinality at most $\mathfrak{a}$ embeds into $M(\mathfrak{a})$. The construction is obtained via the categorical equivalence between the category of all MV-algebras and all unital abelian lattice-ordered groups and Hahn's embedding theorem. The algebra $M(\aleph_0)$ is the first known MV-algebra with respect to which the deductive system for the infinitely-valued Łukasiewicz's propositional logic is strongly complete, and thus provides a strongly adequate model (in the sense of Łoś and Suszko).

We will show (without the construction) that the quasivariety of all GMV-algebras is also generated by a single structure. This result is obtained by using a recent Dvurečenskij-Holland theorem about free products of GMV-algebras.