Varieties of Unital ℓ-Groups and ΨMV-Algebras

Mini-Course, Bowling Green, June, 2009

W. Charles Holland

(Day 1):

References (for general background)

$\longrightarrow$ *Varieties of lattice-ordered groups,* N. R. Reilly, in *Lattice-Ordered Groups, Advances and Techniques*, A. M. W. Glass and W. C. Holland (eds.), Kluwer Academic Publishers, 1989.

Lattice-Ordered Groups, an Introduction, M. Anderson and T. Feil, D. Reidel Pub. Co., 1988.

Theory of Lattice-Ordered Groups, M. Darnel, Marcel Dekker, 1995.

Partially Ordered Groups, A. M. W. Glass, World Scientific, 1999.

The Theory of Lattice-Ordered Groups, V. M. Kopytov and N. Ya. Medvedev, Kluwer Acdemic Publishers, 1994.

Lattice-Ordered Groups, Advances and Techniques, A. M. W. Glass and W. C. Holland (eds.), Kluwer Academic Publishers, 1989, especially Chapters 0,1, and 2.

Algebraic Foundations of Many-valued Reasoning, R. L. O. Cignoli, I. M. L. D'Ottaviano and D. Mundici, Kluwer Academic Publishers, 2000.

1 History

Bettazzi 1890. A commutative archimedean totally ordered group is an ordered subgroup of the reals.

Hölder 1901. An archimedean totally ordered group is an ordered subgroup of the reals (without assuming commutative, and without mentioning Bettazzi).

Hahn 1907. Every commutative totally ordered group embeds in a lexicographic direct product of subgroups of the reals (mentioning Bettazzi, but not Hölder).

Birkhoff 1935 – 1965. General lattice-ordered groups.

Lorenzen 1939. All commutative ℓ-groups are contained in direct products of totally ordered groups.

Conrad 1955 – 2000. General lattice-ordered groups.

Quantum mechanics 1900. Multi-valued logic: Łukasiewich 1920. MV-algebra: Chang 1958.

MV-algebras $\leftrightarrow$ commutative unital ℓ-groups: Mundici 1986 [21].

Pseudo (not necessarily commutative) MV-algebra 2001:
Rachůnek (at my suggestion); Georgescu and Iorgulescu.

Pseudo MV-algebras $\leftrightarrow$ unital ℓ-groups: Dvurečenskij 2002 [5].

2 ΨMV-Algebras

History (my suggestion to Rachůnek).

Definition

A Ψ*MV-algebra* (*pseudo MV-algebra*) is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $1^\sim = 0$; $1^- = 0$,

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x$,
(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y$,
(A8) $(x^{-})^{\sim} = x$.

Exercise. (1) $x \oplus x^{\sim} = 1$. The $x^{\sim}$ is the "right complement" of x.

Proof.

$$\begin{aligned} x \oplus x^{\sim} &\overset{A8}{=} x \oplus (x^{\sim})^{-\sim} \\ &\overset{A2}{=} x \oplus (0 \oplus (x^{\sim})^{-})^{\sim} \\ &\overset{A4}{=} x \oplus (1^{-} \oplus (x^{\sim})^{-})^{\sim} \\ &\overset{\text{def}\odot}{=} x \oplus (x^{\sim} \odot 1) \\ &\overset{A6}{=} 1 \oplus (1^{\sim} \odot x) \\ &\overset{A3}{=} 1. \end{aligned}$$

Exercise (2). $x^{-} \oplus x = 1$. The x^{-} is the "left complement" of x.

If we define $x \leq y$ iff $x^{-} \oplus y = 1$, then $\leq$ is a lattice order.
Proof: ....

Definition An *ℓ-group* is a group and a lattice such that the lattice operations are preserved by the binary group operation: $x(y \vee z) = (xy) \vee (xz)$ and $(y \vee z)x = (yx) \vee (zx)$ (and $x(y \wedge z) = (xy) \wedge (xz)$ and $(y \wedge z)x = (yx) \wedge (zx)$).

Example Let Ω be any totally ordered set and let $\mathrm{Aut}(\Omega)$ be the group of all order-preserving permutations of Ω. Order G by letting $f \leq g$ iff for all $\alpha \in \Omega$, $\alpha f \leq \alpha g$. Then G is an ℓ-group. For most Ω's, $\mathrm{Aut}(\Omega)$ is not commutative.

A ΨMV-algebra is an MV-algebra if and only if $x \oplus y = y \oplus x$ for all $x, y \in M$.

ΨMV-algebra Example

Let G be an ℓ-group and $e \leq u \in G$. Let

$$\Gamma(G, u) = \{g \in G \mid \exists \text{ such that } e \leq g \leq u\}.$$

Let $x \oplus y = (xy) \wedge u$, $x^- = ux^{-1}$, and $x^\sim = x^{-1}u$. Then $\Gamma(G,u)$ is a ΨMV-algebra. Then $\Gamma(G,u)$ is an MV-algebra iff G is commutative.

3 Unital ℓ-groups and ΨMV-algebras are categorically equivalent

Definition A *unital ℓ-group* is (G,u) where G is an ℓ-group (not necessarily abelian), and $e \leq u \in G$ is a *strong unit*. That is, for all $g \in G$ there exists $n \in \mathbb{N}$ such that $u^{-n} \leq g \leq u^n$.

Example. Let G be any ℓ-group, and $e \leq u \in G$. Let

$$H = \{g \in G \mid \exists n \in \mathbb{N} \text{ such that } u^{-n} \leq g \leq u^n\}.$$

Then (H,u) is a unital ℓ-group.

Theorem 3.1 (Mundici, 1986 [21]) *Abelian unital ℓ-groups are categorically equivalent to MV-algebras.*

Theorem 3.2 (Dvurečenskij, 2002 [5]) *Unital ℓ-groups are categorically equivalent to ΨMV-algebras.*

Example

The unital ordered group of real numbers $(\mathbb{R},1)$ with unit 1 is completely determined by the MV-algebra $([0,1],\oplus,^-,0,1)$ with $x \oplus y = (x+y) \wedge 1$ and $x^- = 1-x$.

Outline of proof:

1. First step: extend $[0,1]$ and its arithmetic to $[0,2]$.

On $[0,1]$ we have $u=1$, $x \oplus y = (x+y) \wedge 1$, $x^- = 1-x$, and $\leq$. Then $[0,2] = [0,1] \cup (1+[0,1])$, and $u^* = 1+1$.

For $x,y \in [0,1]$, $x \leq^* 1+y$, $x \leq^* y \Leftrightarrow x \leq y$, and $1+x \leq^* 1+y \Leftrightarrow x \leq y$.

For $x \in [0,1]$, $x^{-*} = 1+x^-$ and $(1+x)^{-*} = x^-$.

In $[0,1]$, for $x \leq y$, define $y \ominus x = (x \oplus y^-)^- = 1-((x+(1-y)) \wedge 1) = 1-(x+1-y) = y-x$. Then $(y \ominus x) \oplus x = ((y-x)+x) \wedge 1 = y \wedge 1 = y$.

Now we look at addition on $[0,2]$. Let $x, y \in [0,1]$ and $x \leq y$. If $x+y \leq 1$, then $x \oplus^* y = x \oplus y = x + y$. If $x + y \geq 1$, then $x \geq 1 - y$ and $x \oplus^* y = x + y = 1 + (x - (1 - y)) = 1 + (x \ominus y)$.

Next, for $x, y \in [0,1]$, $(1 + x) + y = 1 + (x + y) = 1 + (x \oplus^* y)$. So $(1 + x) \oplus^* y = 1 + (x \oplus^* y) = 1 + (x + y)$.

Finally, $(1+x) \oplus^* (1+y) = ((1+x)+(1+y)) \wedge 2 = (2+x+y) \wedge 2 = 2 = u^*$. In particular, $\oplus^*$ is completely determined by $\oplus$

2. We now repeat this process, going from $([0,2], \oplus^*, ^{-*})$ to $([0,4], \oplus^{**}, ^{-**})$, obtaining for each $x, y \in [0,2]$, $x \oplus^{**} y = x \oplus^* y = x \oplus y = x + y$.

Inductively, we have for each n, $([0,2^n], \oplus^{n*}, ^{-n*})$ with for each $x, y \in [0, 2^{n-1}]$, $x + y = x \oplus^{n*} y$.

Finally, for each $0 \leq x, y \in \mathbb{R}$ there exists n such that $x, y \leq 2^{n-1}$, and so $x + y = x \oplus^{n*} y$.

Thus, the positive part of the ordered group of real numbers $\mathbb{R}^+$ is completely determined by $([0,1], \oplus, ^-)$. The entire group is all $x - y$ with $x, y \in \mathbb{R}^+$.

Definitions Let G be an ℓ-group and $e \neq g \in G$. Then there is a (not necessarily unique) convex ℓ-subgroup V of G, maximal wrt $g \notin V$. Then V is a *value* of g. The set G/V of all right cosets Vx has a natural total order with $Vx \leq Vy$ iff $\exists z \in V$ with $x \leq zy$. Then there is an ℓ-homomorphism $\phi : G \to \mathrm{Aut}(G/V)$ with $(Vx)(g\phi) = Vxg$, so $G\phi \subseteq \mathrm{Aut}(G/V)$. Also, letting V^* be the intersection of all convex ℓ-subgroups of G containing V and g, V^* is the *cover* of V, and V^*/V is an interval of G/V. The restriction of ϕ to V^* puts $V^*\phi \subseteq \mathrm{Aut}(V^*/V)$, and $V^*\phi \approx V^*/(\cap_{g \in V^*} V^g)$ (where $V^g = g^{-1}Vg$).

As a consequence,

Theorem 3.3 (Holland, 1963 [9]) *For every ℓ-group G there exists a totally ordered set Ω such that $G \subseteq \mathrm{Aut}(\Omega)$.*

Definition *Variety* = equationally defined class: If Σ is a set of equations with variables $x_1, x_2, \ldots$ and operations of some type of algebra, then the variety defined by Σ is the collection of all algebras G of the given type such that each of the equations in Σ is true for every substitution of elements of G That is, G *satisfies* each equation in Σ. (*Warning!* Not everyone agrees with this definition. They insist on calling these "equational classes".) The variety *generated* by an algebra G is the collection of all algebras which satisfy each of the equations satisfied by G.

Varieties of unital ℓ-groups correspond exactly to varieties of ΨMV-algebras:

Theorem 3.4 (Dvurečenskij and Holland, 2007 [6]) *Let $\mathcal{V}$ be a variety of unital ℓ-groups, and let $\Gamma(\mathcal{V})$ be the collection of all ΨMV-algebras (isomorphic to) $\Gamma(G, u)$ where $(G, u) \in \mathcal{V}$. Then Γ is an isomorphism from the lattice of all varieties of unital ℓ-groups to the lattice of all varieties of ΨMV-algebras.*

Proof. Suppose first that $\mathcal{V}$ is an equational class of unital ℓ-groups. We use the categorical equivalence of ΨMV-algebras and unital ℓ-groups, and we show that $\Gamma(\mathcal{V})$ is a variety by invoking Birkhoff's theorem.

$\Gamma(\mathcal{V})$ is:

1. Closed under subalgebra: Let $\Gamma(H, u)$ be a subalgebra of $\Gamma(G, u)$, and $(G, u) \in \mathcal{V}$. Then (H, u) is a unital ℓ-subgroup of (G, u), and so $(H, u) \in \mathcal{V}$. Hence $\Gamma(H, u) \in \Gamma(\mathcal{V})$.

2. Closed under homomorphic image: Let $\Gamma(G, u) \in \Gamma(\mathcal{V})$ and let $\phi : \Gamma(G, u) \to \Gamma(H, v)$ be an epimorphism. Then $(G, u) \in \mathcal{V}$, and there is an epimorphism $\bar{\phi} : (G, u) \to (H, v)$ such that $\bar{\phi}|_{[e,u]} = \phi$. Then $(H, v) \in \mathcal{V}$, and so $\Gamma(H, v) \in \Gamma(\mathcal{V})$.

3. Closed under product: For any ℓ-group G and any $e \leq u \in G$, let $\langle\langle G; u\rangle\rangle$ denote the convex ℓ-subgroup of G generated by u. That is,

$$\langle\langle G; u\rangle\rangle = \{g \in G \mid \exists n,\, u^{-n} \leq g \leq u^n\}.$$

Then u is a unit of $\langle\langle G; u\rangle\rangle$, and so $(\langle\langle G; u\rangle\rangle, u)$ is a unital ℓ-group. Let Λ be an index set, and for each $\lambda \in \Lambda$ let (G_λ, u_λ) be a unital ℓ-group. Let $\bar{u} \in \Pi G_\lambda$ be such that $\bar{u}(\lambda) = u_\lambda$ for all $\lambda \in \Lambda$. Then $(\langle\langle(\Pi G_\lambda); \bar{u}\rangle\rangle, \bar{u}) \in \mathcal{V}$ because it is clear that each of the equations of $\mathcal{V}$ is satisfied by $(\langle\langle(\Pi G_\lambda); \bar{u}\rangle\rangle, \bar{u})$. And it is also clear that $\Pi(\Gamma(G_\lambda, u_\lambda) = \Gamma(\langle\langle(\Pi G_\lambda); \bar{u}\rangle\rangle, \bar{u})$. Therefore, $\Pi(\Gamma(G_\lambda, u_\lambda)) \in \mathcal{V}$.

Suppose now that $\mathcal{V}_1, \mathcal{V}_2$ are two different equational classes of unital ℓ-groups. We may assume that there is some $(G, u) \in \mathcal{V}_1 \backslash \mathcal{V}_2$. Then $\Gamma(G, u) \in \Gamma(\mathcal{V}_1)$. But $\Gamma(G, u) \notin \Gamma(\mathcal{V}_2)$, because otherwise $\Gamma(G, u)$ is isomorphic to some $\Gamma(H, v)$ with $(H, v) \in \mathcal{V}_2$. But then (G, u) is isomorphic to (H, v), forcing $(G, u) \in \mathcal{V}_2$, a contradiction. Therefore, the mapping is one-to-one.

Now let $\mathcal{W}$ be any variety of ΨMV-algebras. Then $\mathcal{W}$ is defined by a set of equations Σ. For each $(\sigma = \tau) \in \Sigma$, let $(\sigma' = \tau')$ be the equation in the language of unital ℓ-groups obtained from $(\sigma = \tau)$ by replacing $x \oplus y$

with $(xy) \wedge u$, x^- with ux^{-1}, $x^\sim$ with $x^{-1}u$, 0 with e, and 1 with u. Then $\Gamma(G,u) \models (\sigma = \tau)$ iff for all $g_1, \ldots, g_n \in [e,u]$, $\sigma(g_1, \ldots, g_n) = \tau(g_1, \ldots, g_n)$, and this is true iff $[e,u] \models (\sigma' = \tau')$, that is, for all $g_1, \ldots, g_n \in [e,u]$, $\sigma'(g_1, \ldots, g_n) = \tau'(g_1, \ldots, g_n)$. Now let $(\sigma'' = \tau'')$ be the equation obtained by replacing each variable x in $(\sigma' = \tau')$ with $(x \wedge u) \vee e$. Then $[e,u] \models (\sigma' = \tau')$ iff $(G,u) \models (\sigma'' = \tau'')$. Because if $[e,u] \models (\sigma' = \tau')$ and $g_1, \ldots, g_n \in G$, then each $(g_i \wedge u) \vee e \in [e,u]$, so $\sigma''(g_1, \ldots, g_n) = \sigma'((g_1 \wedge u) \vee e, \ldots, (g_n \wedge u) \vee e) = \tau'((g_1 \wedge u) \vee e, \ldots, (g_n \wedge u) \vee e) = \tau''(g_1, \ldots, g_n)$. And if $(G,u) \models (\sigma'' = \tau'')$, and $g_1, \ldots, g_n \in [e,u]$, then each $(g_i \wedge u) \vee e = g_i$, and so $\sigma'(g_1, \ldots, g_n) = \sigma'((g_1 \wedge u) \vee e, \ldots, (g_n \wedge u) \vee e) = \sigma''(g_1, \ldots, g_n) = \tau''(g_1, \ldots, g_n) = \tau'((g_1 \wedge u) \vee e, \ldots, (g_n \wedge u) \vee e) = \tau'(g_1, \ldots, g_n)$. Hence, if $\mathcal{V}$ is the class defined by all equations $(\sigma'' = \tau'')$ with $(\sigma = \tau) \in \Sigma$, then $\Gamma(\mathcal{V}) = \mathcal{W}$. Therefore, the mapping is onto. □

(Day 2):

4 Varieties of ℓ-groups

1. Group equations which hold in ℓ-groups.

Theorem 4.1 *The only group equations which hold in all ℓ-groups are those which hold in all groups.*

Proof. Every free group can be totally ordered (see Darnel's book), and then is an ℓ-group. So if a group equation holds in all ℓ-groups, then it holds in all free groups, and hence also in all groups. □

2. Lattice equations which hold in ℓ-groups.

Theorem 4.2 *The only lattice equations which hold in all ℓ-groups are those which hold in all distributive lattices.*

Proof. Every ℓ-group is a subgroup and a sublattice of $\mathrm{Aut}(\Omega)$ for some totally ordered set Ω (Theorem 3.3). Beause every totally ordered set is a distributive lattice, it is easy to check that for every $f, g, h \in \mathrm{Aut}(\Omega)$ and every $\alpha \in \Omega$,

$$\begin{aligned} \alpha(f \vee (g \wedge h)) &= (\alpha f) \vee ((\alpha g) \wedge (\alpha h)) \\ &= ((\alpha f) \vee \alpha g)) \wedge ((\alpha f) \vee (\alpha h)) \\ &= \alpha((f \vee g) \wedge (f \vee h)). \end{aligned}$$

Hence, $f\vee(g\wedge h)=(f\vee g)\wedge(f\vee h)$; and similarly, $f\wedge(g\vee h)=(f\wedge g)\vee(f\wedge h)$. Therefore every ℓ-group is a distributive lattice. □

Theorem 4.3 *Distributive lattices are the smallest non-trivial variety of lattices (contained in all others).*

General form of an ℓ-group equation: because any ℓ-group is a distributive lattice, and the group product distributes over the lattice operations, and $(x\vee y)^{-1}=x^{-1}\wedge y^{-1}$ and $(x\wedge y)^{-1}=x^{-1}\vee y^{-1}$, every equation has the form:

$$\vee_p \wedge_q \Pi_r x_{pqr} = \vee_l \wedge_m \Pi_n y_{lmn}$$

and so

$$\vee_p \wedge_q \Pi_r x_{pqr}(\vee_l \wedge_m \Pi_n y_{lmn})^{-1} = e$$

and hence equivalently, for certain choice of w_{ijk}, all equations have the form

$$\vee_i \wedge_j \Pi_k w_{ijk} = e.$$

Each of the w_{ijk} is a "group word". That is, an expression of the form $w=w(x_1,x_2,\dots,x_n)=y_1y_2\cdots y_m$ where each

$$y_i \in \{x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}\}.$$

And the expression $\vee_i \wedge_j \Pi_k w_{ijk}$ is an ℓ-group word.

Decidability of the word problem for ℓ-groups

The word problem for ℓ-groups: Is it possible to determine which equations hold in *all* ℓ-groups?

First step: finding the equations which hold in $\mathrm{Aut}(\mathbb{R})$ (or any highly transitive $\mathrm{Aut}(\Omega)$)**:**

Possible action of a group word on $\mathbb{R}$: Let $\alpha\in\mathbb{R}$ and $g_1,g_2,\dots,g_n\in\mathrm{Aut}(\mathbb{R})$.

1.1 $w=x_1$. Then $w(g_1)=g_1$.

$$\alpha g_1 \begin{cases} <\alpha & \text{, or} \\ =\alpha & \text{, or} \\ >\alpha. & \end{cases}$$

$\exists g_1 \in \mathrm{Aut}(\mathbb{R})$ such that $\alpha < \alpha g_1$, so $w(g_1) \neq e$.

1.2 $w = x_1 x_2$. Then $w(g_1, g_2) = g_1 g_2$.

$$\text{If } \alpha g_1 < \alpha \text{, then } \alpha g_1 g_2 = \star : \begin{cases} \star < \alpha g_1 < \alpha & \text{, or} \\ \star = \alpha g_1 < \alpha & \text{, or} \\ \alpha g_1 < \star < \alpha & \text{, or} \\ \alpha g_1 < \star = \alpha & \text{, or} \\ \alpha g_1 < \alpha < \star . \end{cases}$$

Pictures...........

$$\text{If } \alpha g_1 = \alpha \text{, then } \alpha g_1 g_2 = \star : \begin{cases} \star < \alpha g_1 = \alpha & \text{, or} \\ \star = \alpha g_1 = \alpha & \text{, or} \\ \alpha g_1 = \alpha < \star . \end{cases}$$

Pictures....

$$\text{If } \alpha g_1 > \alpha \text{, then } \alpha g_1 g_2 = \star : \begin{cases} \star < \alpha < \alpha g_1 & \text{, or} \\ \star = \alpha < \alpha g_1 & \text{, or} \\ \alpha < \star < \alpha g_1 & \text{, or} \\ \alpha < \alpha g_1 = \star & \text{, or} \\ \alpha < \alpha g_1 < \star . \end{cases}$$

Pictures........

In each case, $\exists g_1, g_2 \in \mathrm{Aut}(\mathbb{R})$ such that $\alpha < \alpha g_1 g_2$, so $w(g_1, g_2) \neq e$.

1.3 $w = x_1 x_2 x_1^{-1}$

$$\text{If } \alpha < \alpha g_1 < \alpha g_1 g_2 \text{ then } \alpha g_1 g_2 g_1^{-1} = \star : \begin{cases} \text{NOT } \star < \alpha < \alpha g_1 < \alpha g_1 g_2 & \text{, or} \\ \text{NOT } \star = \alpha < \alpha g_1 < \alpha g_1 g_2 & \text{, or} \\ \alpha < \star < \alpha g_1 < \alpha g_1 g_2 & \text{, or} \\ \alpha < \alpha g_1 = \star < \alpha g_1 g_2 & \text{, or} \\ \alpha < \alpha g_1 < \star < \alpha g_1 g_2 & \text{, or} \\ \alpha < \alpha g_1 < \alpha g_1 g_2 = \star & \text{, or} \\ \alpha < \alpha g_1 < \alpha g_1 g_2 < \star . \end{cases}$$

Etc.......In each case, $\exists g_1, g_2 \in \mathrm{Aut}(\mathbb{R})$ such that $\alpha < \alpha g_1 g_2 g_1^{-1}$, so $w(g_1, g_2) \neq e$.

And so forth.....

Next, for words of the form $\wedge w_i$ with group words $\{w_i\}$ where $\alpha(\wedge_i w_i) = \wedge_i(\alpha w_i)$. Finally for words of the form $\vee_j(\wedge_i(w_{ji}))$ where

$$\alpha(\vee_j(\wedge_i(w_{ji}))) = \vee_j \wedge_i (\alpha w_{ji}).$$

To see if Aut($\mathbb{R}$) satisfies $\vee_j(\wedge_i(w_{ji})) = e$ with $w_{ij} = w_{ij}(x_1, x_2, \ldots, x_n)$, we need to see if for some choice of $g_1, g_2, \ldots, g_n \in$ Aut($\mathbb{R}$), and for some $\alpha \in R$,

$$\alpha(\vee_j \wedge_i (\alpha w_{ji}(g_1, g_2, \ldots, g_n))) \neq \alpha.$$

For example, consider $w(x_1) = (x_1 \vee e) \wedge (x_1 \wedge e)^{-1}$.

If $\alpha g_1 = \alpha$ then: $\alpha g_1 \vee \alpha e = \alpha \vee \alpha = \alpha = \alpha = \alpha g_1 \wedge \alpha e = \alpha(g_1 \wedge e)$; so $\alpha(g_1 \wedge e)^{-1} = \alpha$ and therefore $\alpha((g_1 \vee e) \wedge (g_1 \wedge e)^{-1}) = \alpha \wedge \alpha = \alpha$.

If $\alpha < \alpha g_1$ then: $\alpha(g_1 \vee e) = \alpha g_1 \vee \alpha e = \alpha g_1$ and $\alpha(g_1 \wedge e) = \alpha g_1 \wedge \alpha e = \alpha$; so $\alpha(g_1 \wedge e)^{-1} = \alpha$ and $\alpha((g_1 \vee e) \wedge (g_1 \wedge e)^{-1})) = \alpha g_1 \wedge \alpha = \alpha$.

If $\alpha g_1 < \alpha$ then: $\alpha(g_1 \vee e) = \alpha g_1 \vee \alpha e = \alpha$ and $\alpha(g_1 \wedge e) = \alpha g_1 \wedge \alpha e = \alpha g_1$. Therefore $\alpha < \alpha(g_1 \wedge e)^{-1}$. Hence $\alpha((g_1 \vee e) \wedge (g_1 \wedge e)^{-1})) = \alpha \wedge \alpha(g \wedge e)^{-1} = \alpha$.

Thus, in any case $\alpha((g_1 \vee e) \wedge (g_1 \wedge e)^{-1})) = \alpha$, so $(g_1 \vee e) \wedge (g_1 \wedge e)^{-1}) = e$.

Therefore, Aut($\mathbb{R}$) satisfies $w(x_1) = e$.

Theorem 4.4 Aut($\mathbb{R}$) *(or any ℓ-group G highly transitive on some Λ) generates the variety of all ℓ-groups.*

Proof. Suppose $w(x_1, x_2, \ldots, x_n) \neq e$ in some ℓ-group G. Then there exist $g_1, g_2, \ldots, g_n \in G$ such that $w(g_1, g_2, \ldots, g_n) = \vee_j \wedge_i (w_{ji}(g_1, g_2, \ldots, g_n))) \neq e$. We can assume that $G \subseteq$ Aut(Ω). For some $\alpha \in \Omega$, $\alpha w(g_1, g_2, \ldots, g_n) \neq \alpha$. There is a finite diagram involving α and each of its images under all parts of the word $w(g_1, g_2, \ldots, g_n)$. This diagram can be embedded into $\mathbb{R}$, and because Aut($\mathbb{R}$) is highly transitive, there are $g_1^*, g_2^*, \ldots, g_n^* \in$ Aut($\mathbb{R}$) which have the same action in the diagram as $g_1, g_2, \ldots, g_n$. Therefore, $\alpha w(g_1^*, g_2^*, \ldots, g_n^*) \neq \alpha$, and so $w(g_1^*, g_2^*, \ldots, g_n^*) \neq e$.

Picture.....

It follows that every equation satisfied by Aut($\mathbb{R}$) is satisfied by all ℓ-groups, and so Aut($\mathbb{R}$) generates the variety of all ℓ-groups.

Therefore,

Theorem 4.5 (Holland and McCleary, 1970 [14]) *The word problem for free ℓ-groups (that is, for the class of all ℓ-groups) is solvable.*

Theorem 4.6 (Weinberg, 1963 [24]) *The Abelian variety of ℓ-groups is generated by $\mathbb{Z}$, and so it is the unique minimal non-trival variety of ℓ-groups.*

Definition If $G \subseteq \mathrm{Aut}(\Omega)$, then G is *primitive* if G is transitive on Ω and there is no non-trivial G-congruence on Ω. That is, no equivalence relation $\approx$ such that

(i) $\alpha \approx \beta$ and $g \in G$ implies $\alpha g \approx \beta g$; and

(ii) $\alpha < \beta < \gamma$ and $\alpha \approx \gamma$ implies $\alpha \approx \beta$

except for the relations $\alpha \approx \beta \Leftrightarrow \alpha = \beta$, and $\forall \alpha, \beta, \alpha \approx \beta$.

For example, for any ℓ-group G, if $e \neq g \in G$ and V is a value of g with cover V^*, then the induced image $V^*\phi \subseteq \mathrm{Aut}(V^*/V)$ is primitive. $V^*\phi$ is called a *component* of G

McCleary's Trichotomy ([19]) There are just three types of primitive ℓ-permutation groups $G \subseteq \mathrm{Aut}(\Omega)$:

1. G is (isomorphic to) an ℓ-subgroup of the real numbers, permuting itself by translation;

2. G is a *highly transitive* ℓ-permutation group. That is, for any two finite subsets of Ω of the same size, $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $\beta_1 < \beta_2 < \cdots < \beta_n$, there exists $g \in G$ such that for each i, $\alpha_i g = \beta_i$;

3. G is *periodic*; that is, Ω is dense in itself, and lettting $\bar{\Omega}$ be the dedekind completion of Ω, there is a natural embedding $G \subseteq \mathrm{Aut}(\bar{\Omega})$ and an element $t \in \mathrm{Aut}(\bar{\Omega})$ such that for any $\alpha \in \Omega$, $\{\alpha t^n \mid n \in \mathbb{Z}\}$ has no upper or lower bounds, and for all $g \in G$, $gt = tg$; and if $h \in \mathrm{Aut}(\bar{\Omega})$ and for all $g \in G$, $hg = gh$, then for some n, $h = t^n$. Then t is the *period* of G. In this case, G is *locally highly transitive*. That is, for any $\alpha \in \Omega$, G is highly transitive on the interval $(\alpha, \alpha t)$. Also, G is not commutative.

Thus, these are the only three types of components of an ℓ-group. If every value in G is normal in its cover, then G is *normal valued*. The collection of all normal-valued ℓ-groups is denoted by $\mathcal{N}$, and in $\mathcal{N}$, every component is a subgroup of the real numbers.

Theorem 4.7 (Wolfenstein,1968 [25]) *$\mathcal{N}$ is a variety of ℓ-groups, defined by the "equation": for all $e \leq x, y$, $xy \leq y^2x^2$ (equivalent to $|x||y| \wedge |y|^2|x|^2 = |x||y|$.)*

Example Every totally ordered group is normal valued. But any ℓ-permutation group which is highly transitive, or even locally highly transitive fails Weinberg's equation, and so is not normal valued.

Theorem 4.8 (Holland, 1976 [10]) *$\mathcal{N}$ is the unique maximal proper variety of ℓ-groups.*

Proof. $\mathcal{N}$ is proper; for example, Aut($\mathbb{R}$) is highly transitive, and so Aut($\mathbb{R}$) $\notin \mathcal{N}$. And if $G \notin \mathcal{N}$, then some $g \in G$ has a value V which is not normal in its cover V^*. Then the primitive component $V^*\phi$ is not commutative, and so is either highly transitive or locally highly transitive. Then $V^*\phi$ generates the variety of all ℓ-groups. But $V^*\phi$ must lie in the variety generated by G, and so the variety of G is all ℓ-groups. □

(Day 3):

Covering Layer of the Abelian Variety

Because the Abelian variety of ℓ-groups $\mathcal{A}$ is defined by a single equation, every variety which properly contains $\mathcal{A}$ contains a cover of $\mathcal{A}$. Thus, $\mathcal{A}$ has a *covering layer* of varieties.

Examples of covers of $\mathcal{A}$

The Scrimger varieties (1975 [23]). For each prime number p, let $\mathcal{S}_p$ be the variety generated by $(\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_n\rangle) \overleftarrow{\rtimes} \langle b\rangle$, where for $1 \leq i < n$, $b^{-1}a_ib = a_{i+1}$ and $b^{-1}a_{i+1}b = a_1$.

The Medvedev varieties (1977 [20]). Let $\mathcal{M}^+$ be the variety generated by

$$(\overleftarrow{\Sigma}_{i\in\mathbb{Z}}\ \langle a_i\rangle) \overleftarrow{\rtimes} \langle b\rangle$$

where $b^{-1}a_ib = a_{i+1}$.

Let $\mathcal{M}^-$ be the variety generated by

$$(\overleftarrow{\Sigma}_{i\in\mathbb{Z}}\ \langle a_i\rangle) \overleftarrow{\rtimes} \langle b\rangle$$

where $b^{-1}a_ib = a_{i-1}$.

Let $\mathcal{M}_0$ be the variety generated by

$$(\langle a_1 \rangle \overleftarrow{\times} \langle a_2 \rangle) \overleftarrow{\rtimes} \langle b \rangle$$

where $ba_1 = a_1 b$ and $b^{-1} a_2 b = a_1 a_2$.

Let G be a totally ordered group and $x, y \in G$. Then $x \ll y$ means that for all $n \in \mathbb{N}$, $e < x^n < y$.

The Feil-Bergman-Kopytov varieties (1980 [7], 1984 [1], 1985 [17]). Let $F = \langle a, b \rangle$ be the free group on $\{a, b\}$. Then F can be made into a totally ordered group F_+ such that if $x \ll y$ then $x \ll y^{-1} x y$. Let $\mathcal{F}_+$ be the variety generated by F_+.

Similarly, F can be made into a totally ordered group F_- such that if $x \ll y$ then $(y^{-1} x y) \ll x$. Let $\mathcal{F}_-$ be the variety generated by F_-.

The Holland-Medvedev varieties (1994 [15]). Let $F = \langle a, b \rangle$ be the free group on generators $\{a, b\}$ and let $s = (s_1, s_2, \ldots)$ be a sequence of ± 1's. There is a total order of F with the following properties:

(1) $e < a \ll b$; let $g_0 = a$ and $h_0 = b$.

(2) If $s_1 = +1$ then $a \ll a^b$ and if $s_1 = -1$ then $a^b \ll a$; in either case, let $g_1 = a \wedge a^b$ and $h_1 = a \vee a^b$.

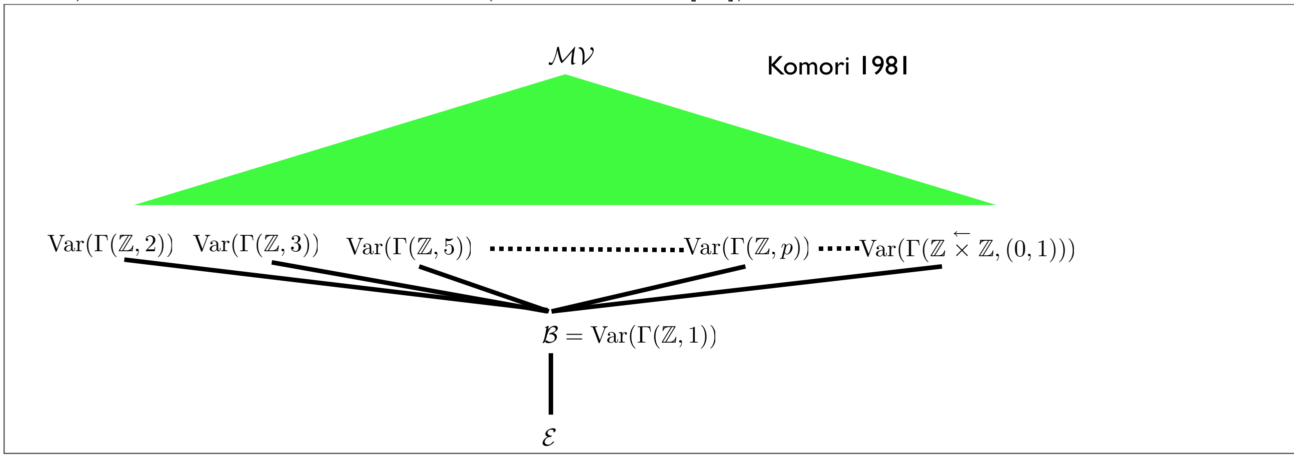
(3) For every $i \in \mathbb{N}$, $g_i \ll h_i$; if $s_{i+1} = +1$ then $h_i \ll h_i^{g_i}$ and if $s_{i+1} = -1$ then $h_i^{g_i} \ll h_i$; in either case, let $g_{i+1} = h_i \wedge h_i^{g_i}$ and $h_{i+1} = h_i \vee h_i^{g_i}$.

Let F_s be the group F with the described total order and let $\mathcal{F}_s$ be the variety generated by F_s. Then $\mathcal{F}_s$ is a cover of $\mathcal{A}$. and although some different sequences can produce the same variety (eg., $(+1, -1, +1, -1, \ldots)$ and $(-1, +1, -1, +1, \ldots)$), there are continuum many different $\mathcal{F}_s$'s.

There may be other covers of $\mathcal{A}$, but so far (in the last 15 years), none have been found.

5 Varieties of unital ℓ-groups

Varieties of Abelian uℓ-groups, equivalent to varieties of MV-algebras (Theorem 3.4): Completely determined by (Komori, 1981 [16])



Boolean $= \mathcal{B}$, generated by $(\mathbb{Z}, 1)$.

Every variety of Abelian unital ℓ-groups is the join of a set of varieties of the following types:
$\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \ldots, \mathcal{K}_n, \ldots$
$\mathrm{Var}(\Gamma(\mathbb{Z}, n)) \leftrightarrow \mathcal{K}_n = \mathrm{Var}_{u\ell}(\mathbb{Z}, n)$ is the uℓ-variety generated by $(\mathbb{Z}, n)$;
and $\mathcal{K}_{\infty,1}, \mathcal{K}_{\infty,2}, \ldots$
$\mathrm{Var}(\Gamma(\mathbb{Z} \overset{\leftarrow}{\times} \mathbb{Z}, (0, n))) \leftrightarrow \mathcal{K}_{\infty,n} = \mathrm{Var}_{u\ell}(\mathbb{Z} \overset{\leftarrow}{\times} \mathbb{Z}, (0, n))$ is the uℓ-variety generated by $(\mathbb{Z} \overset{\leftarrow}{\times} \mathbb{Z}, (0, n))$.

Theorem 5.1 (Chang's completeness theorem (1958 [2])

$$\mathcal{MV} = \mathrm{Var}(\Gamma(\mathbb{R}, 1)) \leftrightarrow \mathcal{A}_{u\ell}.$$

Theorem 5.2 (ΨMV completeness theorem (2007 [6]) *Let* $u \in \mathrm{Aut}(\mathbb{R})$ *be the translation* $\alpha u = \alpha + 1$*, and*

$$\mathrm{BAut}(\mathbb{R}) = \{g \in \mathrm{Aut}(\mathbb{R})) \mid \exists n \in \mathbb{N},\ \ u^{-n} \leq g \leq u^n\}.$$

Then the equations satisfied by $\Gamma(\mathrm{BAut}(\mathbb{R}), u)$ *are just the equations satisfied by all GMV-algebras. So* $\mathrm{Var}(\mathrm{BAut}(\mathbb{R}), u) \leftrightarrow$ *all uℓ-groups.*

Theorem 5.3 *Every proper variety of unital ℓ-groups contains $\mathcal{B}$ = the Boolean variety.*

Proof. If (G, u) is a unital ℓ-group and $u \neq e$, then a subalgebra of (G, u) is $(\langle u \rangle, u) \approx (\mathbb{Z}, 1)$. □

Bottom varieties: covers of Boolean

Theorem 5.4 *The Boolean variety $\mathcal{B}$ is defined by the equation;*

$$((x \vee e) \wedge u)^2 \wedge u = (x \vee e) \wedge u.$$

(This corresponds to the GMV-equation $x \oplus x = x$.)

Proof. $(\mathbb{Z}, 1)$ clearly satisfies the equation. Conversely, suppose (G, u) satisfies the equation. We may assume that (G, u) is subdirectly irreducible and $G \neq \{e\}$. Then G is a transitive ℓ-subgroup of some $\mathrm{Aut}(\Omega)$. Let $\alpha \in \Omega$ and $e \leq g \leq u \in G$. Then $\{\alpha u^n \mid n \in \mathbb{Z}\}$ is unbounded above and below in Ω. Assume that $\alpha \neq \alpha g \neq \alpha u$. Then $\alpha < \alpha g < \alpha u$. Then $\alpha g < \alpha g^2 \wedge \alpha u$, a contradiction. Therefore, for all $\alpha \in \Omega$, either $\alpha g = \alpha$ or $\alpha g = \alpha u$.

Suppose that $g \neq e$. Then for some α, $\alpha < \alpha g = \alpha u$. Then $\alpha g < (\alpha g)g$, so also $\alpha g^2 = (\alpha g)g = (\alpha g)u = \alpha u^2$. Similarly, for all $n \in \mathbb{Z}$, $\alpha g^n = \alpha u^n$. But then $\{\alpha g^n \mid n \in \mathbb{Z}\}$ is unbounded, so for every $\beta \in \Omega$ there exists $n \in \mathbb{Z}$ with $\alpha g^n \leq \beta < \alpha g^{n+1}$. Then $\beta < \alpha g^{n+1} \leq \beta g$, so $\beta \neq \beta g$, and therefore $\beta g = \beta u$. It follows that $g = u$.

Thus, in G, the interval $[e, u] = \{e, u\}$, and since G is determined by this interval, $(G, u) = (\langle u \rangle, u) \approx (\mathbb{Z}, 1)$. □

Komori varieties

$\mathcal{K}_p = \mathrm{Var}(\mathbb{Z}, p)$ for each prime p; and $\mathcal{K}_{\infty,1} = \mathrm{Var}(\mathbb{Z} \overleftarrow{\times} \mathbb{Z}, (0, 1))$.

Medvedev varieties

The first two noncommutative varieties of unital ℓ–groups covering $\mathcal{B}$ were described in [11], and were based upon two of the three Medvedev varieties of ℓ–groups which cover the variety of abelian ℓ–groups [20].

$\mathcal{M}^+$ is the ℓ-group variety generated by

$$(\overleftarrow{\Sigma}_{i \in \mathbb{Z}} \langle a_i \rangle) \overleftarrow{\rtimes} \langle b \rangle$$

where $b^{-1} a_i b = a_{i+1}$. Let $\mathcal{M}^+_{u\ell} = \mathrm{Var}_{u\ell}((\overleftarrow{\Sigma}_{i \in \mathbb{Z}} \langle a_i \rangle) \overleftarrow{\rtimes} \langle b \rangle, (\bar{0}, b))$.

$\mathcal{M}^-$ is the ℓ-group variety generated by

$$(\overleftarrow{\Sigma}_{i\in\mathbb{Z}}\ \langle a_i\rangle)\ \overleftarrow{\rtimes}\ \langle b\rangle$$

where $b^{-1}a_ib = a_{i-1}$. Let $\mathcal{M}^-_{u\ell} = \mathrm{Var}_{u\ell}((\overleftarrow{\Sigma}_{i\in\mathbb{Z}}\ \langle a_i\rangle)\ \overleftarrow{\rtimes}\ \langle b\rangle, (\bar{0}, b))$.

Then $\mathcal{M}^+_{u\ell}$ and $\mathcal{M}^-_{u\ell}$ cover $\mathcal{B}$.

It was also shown that $\mathcal{M}^-_{u\ell} \neq \mathcal{M}^+_{u\ell}$.

Feil varieties For each real number $0 < \alpha \neq 1$, let C_α be the subgroup of $\mathbb{R}$ generated by $\{\alpha^i \mid i \in \mathbb{Z}\}$, and let $F_\alpha = C_\alpha \overleftarrow{\rtimes}_\phi \mathbb{Z}$ where for $c \in C$, $c\phi = c \cdot \alpha$. Let $\mathcal{F}_\alpha = \mathrm{Var}_{u\ell}(F_\alpha, (\bar{0}, 1))$. (Note: this is not $\mathcal{F}_s$ where s is a binary sequence.)

Each $\mathcal{F}_\alpha$ covers $\mathcal{B}$, and if $\alpha \neq \beta$ then $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ ([12]).

Theorem 5.5 (Darnel and Holland 2009 [3]) *The only solvable covers of the Boolean variety of uℓ-groups are $\mathcal{K}_p$ for p a prime, $\mathcal{K}_{\infty,1}$, $\mathcal{M}^+_{u\ell}$, $\mathcal{M}^-_{u\ell}$, and $\mathcal{F}_\alpha$ for $\alpha \in \mathbb{R}$ with $0 < \alpha \neq 1$.*

There surely must be some *non*-solvable covers of $\mathcal{B}$, but none are known.

(Day 4):

6 Top varieties

Definition If V is a value of the unit u in a uℓ-group, then the cover of V must be $V^* = G$. Let ϕ be the natural homomorphism $\phi : (G, u) \to \mathrm{Aut}(G/V)$. Then $G\phi \approx G/(\cap_{g\in G} V^g)$, and $(G\phi, u\phi)$ is a *top component* of (G, u), and is a primitive ℓ-automorphism group on the totally ordered set G/V.

If $\mathcal{V}$ is a variety of uℓ-groups then *top* $\mathcal{V}$ is the collection $\hat{\mathcal{V}}$ of all uℓ-groups in which every top component belongs to $\mathcal{V}$. Clearly, if $\mathcal{V} \subseteq \mathcal{W}$ then $\hat{\mathcal{V}} \subseteq \hat{\mathcal{W}}$.

Theorem 6.1 (Dvurečenskij and Holland 2007 [6]) *For every variety $\mathcal{V}$ of uℓ-groups, $\hat{\mathcal{V}}$ is a variety of uℓ-groups, and $\mathcal{V} \subseteq \hat{\mathcal{V}}$.*

Proof (outline). These are equivalent:

1. For every value V of u, the component $(G/K(V), K(V)u)$ satisfies $w = e$ (where $K(V)$ is the intersection of all conjugates of V);

2. (G, u) satisfies all "equations" of the form $|w_1 w_2 \cdots w_n| \leq u$, where each w_i is a conjugate of w. □

Definition If $\mathcal{V}$ is a variety of uℓ-groups, then $\hat{\mathcal{V}}$ is a *top variety.*

Each top component of any (G, u) is primitive, so by McCleary's Trichotomy, must be either:

1. A subgroup of the reals with some choice of unit,
2. Highly transitive, or
3. Periodic.

Examples of Top Varieties

In what follows, if $\mathcal{V}$ is a variety of ℓ-groups, then $\mathcal{V}_{u\ell}$ will denote the collection of all unital ℓ-groups in $\mathcal{V}$. Then $\mathcal{V}_{u\ell}$ is the variety of uℓ-groups defined by the same equations which define $\mathcal{V}$ (and don't contain u).

1. Top Abelian $= \hat{\mathcal{A}}_{u\ell}$. If each top component of (G, u) is Abelian, then each is isomorphic to a subalgebra of $(\mathbb{R}, 1)$.

2. Top Boolean $= \hat{\mathcal{B}}$. Each top component is isomorphic to $(\mathbb{Z}, 1)$.

3. Top Normal-valued $= \hat{\mathcal{N}}_{u\ell}$. If each top component of (G, u) is normal valued, then since neither highly transitive nor periodic is normal valued, each is isomorphic to a subalgebra of $(\mathbb{R}, 1)$. Hence (G, u) is top Abelian. And since $\mathcal{A} \subseteq \mathcal{N}$, $\hat{\mathcal{A}}_{u\ell} \subseteq \hat{\mathcal{N}}_{u\ell}$. Therefore $\hat{A}_{u\ell} = \hat{\mathcal{N}}_{u\ell}$.

4. If $\mathcal{V}$ is a variety of ℓ-groups, and if $\mathcal{V}$ is proper, then $\mathcal{A} \subseteq \mathcal{V} \subseteq \mathcal{N}$. Therefore, $\hat{\mathcal{V}}_{u\ell} = \hat{\mathcal{A}}_{u\ell}$.

5. If each top component of (G, u) is Abelian, then each will be an ℓ-subgroup of the reals with some choice of unit. Thus, each will be either $(\mathbb{Z}, n)$ or dense. If one is dense, the the top variety is $\hat{\mathcal{A}}_{u\ell}$. In the other cases, it will be the join of the top varieties generated by various $(\mathbb{Z}, n)$. In particular, it will not include $\mathcal{K}_{\infty,1}$.

6. (See Theorem 4.4) If some top component of (G, u) is highly transitive, then the top variety generated by (G, u) is $\mathcal{L}_{u\ell} =$ *all* uℓ-groups.

7. In $\mathrm{Aut}(\mathbb{R})$, for each $r \in \mathbb{R}$ let t_r be translation by r; that is, for all $\alpha \in \mathbb{R}$, $\alpha t_r = \alpha + r$. Let $G = \{g \in \mathrm{Aut}(\mathbb{R}) \mid gt_1 = t_1 g\}$. In particular, $t_1^2 = t_2$. Then G is a primitive periodic ℓ-group. Each of t_1 and t_2 is a unit of G, and the variety generated by (G, t_2) is contained in the variety generated by (G, t_1). Proof: The mapping $\phi : G \to G$ defined by $\alpha(g\phi) = ((2\alpha)g)/2$ is an embedding of the ℓ-group G into G, and $t_2\phi = t_1$, so $\phi : (G, t_2) \to (G, t_1)$ is a uℓ-group embedding.

Definitions

For any uℓ-group (G, u) let $\mathrm{TopComp}(G, u)$ be the set of all top components of (G, u).

For any set $\mathcal{W}$ of uℓ-equations (of the form $w = w(x_1, x_2, \ldots, x_n) = e$), let $\mathrm{V}(\mathcal{W})$ be the variety of uℓ-groups which satisfy all members of $\mathcal{W}$

Let $\hat{\mathrm{V}}(\mathcal{W})$ be the top variety corresponding to $\mathrm{V}(\mathcal{W})$ that is, all uℓ-groups (G, u) such that $\mathrm{TopComp}(G, u) \subseteq \mathrm{V}(\mathcal{W})$.

Let $\mathrm{P}(\mathcal{W})$ be all primitive uℓ-groups which satisfy all members of $\mathcal{W}$; $\mathrm{P}(\mathcal{W})$ is the *primitive variety* defined by $\mathcal{W}$. For a primitive uℓ-group (G, u), let $\mathrm{P}(G, u)$ be the primitive variety generated by (G, u); that is, if $\mathcal{W}$ is the set of all equations satisfied by (G, u), then $\mathrm{P}(G, u) = \mathrm{P}(\mathcal{W})$.

Let $\mathcal{W}^*$ be the set of all equations of the form $|w_1 w_2 \cdots w_n| \leq u$ where each w_i is a conjugate of some w with $(w = e) \in \mathcal{W}$.

Lemma 6.2 *For any set $\mathcal{W}$ of equations and any uℓ-group (G, u), TopComp$(G, u) \subseteq \mathrm{P}(\mathcal{W})$ iff $(G, u) \in \mathrm{V}(\mathcal{W}^*)$.*

7 Primitive Varieties

Each primitive variety is the join of the *principal* primitive varieties contained in it; that is, the primitive varieties generated by a single primitive uℓ-group. Therefore, what remains is to investigate these principal primitive varieties, and how they are related to each other.

In what follows, let (G, u) be primitive. There are just three types.

1. Highly transitive.

Theorem 7.1 *If (G, u) is highly transitive, then $V(G, u) = \mathcal{L}_{u\ell}$ = all $u\ell$-groups, so $P(G, u)$ is all primitive $u\ell$-groups.*

Proof. By Theorem 4.4, (G, u) generates the uℓ-variety of all uℓ-groups. Therefore P(G, u) is all primitive uℓ-groups.

2. Abelian.

If (G, u) is Abelian then G is an ℓ-subgroup of the totally ordered group of real numbers.

If G is dense, then by Komori's theorem, V(G, u) is *all* Abelian uℓ-groups, and so P(G, u) is all ℓ-subgroups of the totally ordered group of real numbers with arbitrary choice of unit.

If G is not dense then $G \approx \mathbb{Z}$ and so $(G, u) \approx (\mathbb{Z}, n)$ for some positive integer n. Then P$(G, u) = \{(\mathbb{Z}, m) \mid m|n\}$, a finite set. And P$(\mathbb{Z}, m) \subseteq$ P$(\mathbb{Z}, n)$ iff $m|n$.

(Day 5):

3. Periodic.

Now assume that (G, u) is primitive periodic.

Theorem 7.2 *Let*

$$G = \{g \in \text{Aut}(\mathbb{R}) \mid gt_1 = t_1 g\}.$$

Then every variety generated by a primitive periodic $u\ell$-group is also generated by (G, u) for some $u \in G$ such that for all $\alpha \in \mathbb{R}$, $\alpha < \alpha u$.

Thus, in what follows we assume that G_t is the ℓ-subgroup of Aut$(\mathbb{R})$ consisting of all g such that $gt_1 = t_1 g$, and we need only examine P(G_t, u) for various choices of u.

To proveTheorem 7.2 we need the following lemmas.

Lemma 7.3 *For any $u\ell$-group (H, u), the variety generated by (H, u) is the same as the variety generated by a certain countable subalgebra of (H, u).*

Proof. There are only a countable number of equations. For each equation which is violated by (H, u) there is a finite set of elements which violate the equation. So there is a countable set S of elements of H such that each equation violated by (H, u) is violated by a subset of S. Therefore, the subalgebra generated by S is countable and generates the same variety as (H, u).

Lemma 7.4 *If $g \in \mathrm{Aut}(\mathbb{R})$ and for all $\alpha \in \mathbb{R}$, $\alpha < \alpha g$, then there exists an automorphism ϕ of $\mathrm{Aut}(\mathbb{R})$ such that $g\phi = t_1$.*

Proof. Let $0 \in \mathbb{R}$. There exists an isomorphism of the ordered sets $\psi : [0, 0g) \to [0, 0t_1]$. Extend ψ to $[0g, 0g^2)$ by $\alpha\psi = \alpha g^{-1}\psi t_1$. Continue to all of $\mathbb{R}$ so that for each $n \in \mathbb{Z}$, if $0g^n \leq \alpha < 0g^{n+1}$, then $\alpha\psi = \alpha g^{-1}\psi t_1$. Then $\psi \in \mathrm{Aut}(\mathbb{R})$. Now define $\phi : \mathrm{Aut}(R) \to \mathrm{Aut}(\mathbb{R})$ by $f\phi = \psi^{-1} f \psi$. Then $g\phi = \psi^{-1} g \psi = \psi^{-1} g(g^{-1}\psi t_1) = t_1$.

Lemma 7.5 *If $H \subseteq \mathrm{Aut}(\Omega)$ is a countable primitive periodic ℓ-group on Ω with period p then there exists an embedding $\phi : H \to \mathrm{Aut}(\mathbb{R})$ onto an ℓ-subgroup with period t_1.*

Proof. Then Ω is a countable set which is dense in itself, and therefore $\Omega \approx \mathbb{Q}$. Then every $g \in H$ as an automorphism of $\mathbb{Q}$ can be uniquely extended to an automorphism of $\mathbb{R}$. The period p will have the property that for every $\alpha \in \mathbb{R}$, $\alpha < \alpha p$ (otherwise, for some α, $\alpha p = \alpha$, and then for all $g \in H$, also $\alpha g = \alpha$. But since H was transitive on Ω, it is also transitive on $\mathbb{Q}$, so it cannot be that every g fixes α).

It now follows from the previous Lemmas that there is an isomorphism of $H \to \mathrm{Aut}(\mathbb{R})$ which sends $p \mapsto t_1$.

Now, to prove the theorem. Let (H, u) be a primitive periodic uℓ-group and let $\mathcal{V}$ be the variety it generates. By Lemma 7.3 we may assume that H is countable, and so by Lemma 7.5 we may assume that H is an ℓ-subgroup of $\mathrm{Aut}(\mathbb{R})$ with period t_1. Since G_t is the group of all elements of $\mathrm{Aut}(\mathbb{R})$ which commute with t_1, then H is an ℓ-subgroup of G_t. Let $\mathcal{V}^*$ be the variety of uℓ-groups generated by (G_t, u). Then clearly $\mathcal{V} \subseteq \mathcal{V}^*$. But because of the local high transitivity of G_t, any equation which is denied by (H, u) is also denied by (G_t, u). Therefore, $\mathcal{V} = \mathcal{V}^*$. □

For the following five theorems, see [13].

Theorem 7.6 *$P(G, t_m) \subseteq \mathrm{P}(G_t, t_n)$ iff $n|m$.*

Proof. There exists an embedding $\phi : (G_t, t_m) \hookrightarrow (G_t, t_n)$ such that $t_m\phi = t_n$ and $t_1\phi = t_{n/m}$. (See Example 7, which is a special case of this.) □

Theorem 7.7 *$P(G_t, t_{1/p}) \subseteq \mathrm{P}(G_t, t_{1/q})$ iff $p|q$.*

Theorem 7.8 *$P(G_t, t_{1/p}) = \mathrm{P}(xu^p = u^p x)$. That is, $P(G_t, t_{1/p})$ is defined by the single equation $xu^p = u^p x$.*

Theorem 7.9 *$P(G_t, t_{m/p}) \subseteq \mathrm{P}(G_t, t_{n/q})$ iff $p|q$ and $n|m$.*

Theorem 7.10 *For $n \geq 2$, $P(G_t, t_{n/q}) = \mathrm{P}((xy^{-1} \wedge y^n u^{-q} \wedge u^q x^{-n}) \vee e = e)$.*

Note: This equation is equivalent to "For all $x, y \in G_t$ and all $\alpha \in \mathbb{R}$, either $\alpha x \leq \alpha y$ or $\alpha y^n \leq \alpha u^q$ or $\alpha u^q \leq \alpha x^n$. "

For the following four theorems, see [4].

Theorem 7.11 *The intersection of any infinite collection of $P(G_t, t_n)$'s is $P(\mathbb{R}, 1)$, the set of all Abelian primitive $u\ell$-groups.*

Theorem 7.12 *The join of any infinite collection of $P(G_t, t_{1/n})$ is the variety of all primitive $u\ell$-groups.*

Theorem 7.13 *Every $P(G_t, u)$ is proper, that is, not equal to all primitive $u\ell$-groups.*

Theorem 7.14 *For positive real numbers r, s with $r \neq s$, $P(G_t, t_r) \neq \mathrm{P}(G_t, t_s)$. Therefore, there are continuum many primitive varieties.*

Theorem 7.15 (Darnel and Holland [4]) *The intersection of all $P(G_t, u)$ contains the Abelian variety $\mathcal{A}_{u\ell}$, and is contained in the symmetric and top Abelian varieties.*

Some open questions

1. What are the relations between $\mathrm{P}(G_t, t_r)$ and $\mathrm{P}(G_t, t_s)$ when at least one of r, s is irrational?

2. What about $\mathrm{P}(G_t, u)$ when u is not a translation?

3. Is the intersection of all the primitive periodic varieties equal to $\mathcal{A}$? Or equal to the intersection of the symmetric and top Abelian varieties? (See Theorem 7.15.)

4. Are there any *non*-solvable covers of $\mathcal{B}$? They would probably be generated by totally ordered uℓ-groups. The non-solvable covers of the Abelian variety of ℓ-groups are all generated by totally ordered groups.

Other observations. There are continuum many $g \in G_t$ such that $g^2 = t_1$, and for each of them $\mathrm{P}(G_t, g) = \mathrm{P}(G_t, t_{1/2})$.

References

[1] G. M. Bergman, *Specially ordered groups*, Comm. Algebra **12**(1984), 2315–2333.

[2] C.C. Chang, *Algebraic analysis of many valued logics,* Trans. Amer. Math. Soc. **88** (1958), 467–490.

[3] M. R. Darnel and W. C. Holland, *Solvable covers of the Boolan variety of unital ℓ-groups*, (to appear in Algebra Universalis).

[4] M. R. Darnel and W. C. Holland, (in progress).

[5] A. Dvurečenskij, *Pseudo MV-algebras are intervals in ℓ-groups*, J. Austral. Math. Soc. **72** (2002), 427–445.

[6] A. Dvurečenskij and W. C. Holland, *Top varities of generalized MV-algebras and unital lattice-ordered groups*, Comm. Algebra **35** (2007), 3370–3390.

[7] T. Feil, *Varieties of Representable Lattice-Ordered Groups*, Doctoral thesis, Bowling Green State University, 1980.

[8] G. Georgescu and A. Iorgulescu, *Pseudo MV algebras*, Mult.-Valued Log. **6** (2001), 95–135.

[9] W. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J. **10** (1963), 399-408.

[10] W.C. Holland, *The largest proper variety of lattice-ordered groups*, Proc. Amer. Math. Soc. **16** (1976), 25–28.

[11] W. C. Holland, *Small varieties of lattice-ordered groups and MV-algebras*, in Contributions to General Algebra, Heyn, Klagenfurt, **16** (2005), 107–114.

[12] W. C. Holland, *Covers of the Boolean variety in the lattice of varieties of unital lattice-ordered groups and GMV-algebras*, In: Selected Questions of Algebra, Collection of papers dedicated to the memory of N. Ya. Medvedev (Bayanova N., eds.), Altai State University Barnaul, Barnaul, 2007, pp. 208–217.

[13] W. C. Holland, *Continuum Many Top Varieties of GMV-Algebras and Unital ?-Groups*, (to appear in J. Algebra).

[14] W. C. Holland and S. H. McCleary, *Solvability of the word problem for free lattice-ordered groups*, Houston J. Math. **5** (1970), 99–105.

[15] W.C. Holland and N. Ya. Medvedev, *A very large class of small varieties of lattice-ordered groups*, Comm. Algebra **22** (1994), 551–578.

[16] Y. Komori, *Super Łukasiewicz propositional logics*, Nagoya Math. J. **84** (1981), 119–133.

[17] V.M. Kopytov, *Non-Abelian varieties of lattice-ordered groups in which every solvable ℓ-group is Abelian*, Mat. Sb. N. Ser. **126(168)** (1985), 247–266.

[18] J. Łukasiewicz, *O logice trójwartościowej,* Ruch Filozoficzny **6** (1920), 170–171. Translated by O. Wojtasiewicz, in: L. Borkowski, *Selected Works of J. Łukasiewicz,* North-Holland, Amsterdam, 1970.

[19] S. H. McCleary, *o-primitive ordered permutation groups*, Pacific J. Math. **49** (1973), 417-424.

[20] N.Ya. Medvedev, *The lattices of varieties of lattice-ordered groups and Lie algebras*, Algebra Logic **16** (1978), 27–31; (Algebra i Logika **16** (1977), 40–45).

[21] D. Mundici, *Interpretation of AFC*-algebras in Łukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15–63.

[22] J. Rachůnek, *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255–273.

[23] E.B. Scrimger, *A large class of small varieties of lattice-ordered groups*, Proc. Amer. Math. Soc. **51** (1975), 301–306.

[24] E.C. Weinberg, *Free lattice-ordered Abelian groups*, Math. Ann. **151** (1963), 187–199.

[25] S. Wolfenstein, *Valuers normals dans un groupe reticule, Accademia Nazionale dei Lincei*, **44**(1968), 337–342.