Lemma 1 (Little Lemma) Suppose $P \cdot Q \cdot R$. Then $P$ and $R$ are on opposite sides of any line through $Q$ (except for $PQ$). Furthermore, $P$ and $Q$ are on the same side of any line through $R$ (except for $PR$).

Theorem 1 [The Crossbar Theorem] Suppose $D$ is in the interior of $\angle ABC$. Then $AC \cap BD \neq \emptyset$.

Proof: Suppose by way of contradiction that $AC \cap BD = \emptyset$.

Recall that $D$ in the interior of $\angle ABC$ means that $D$ and $A$ are on the same side of $BC$ and $D$ and $C$ are on the same side of $AB$. By Axiom K there is a point $E$ for which $D \rightarrow E \rightarrow C$. By theorem 10, it follows that $E$ is in the interior of $\angle ABC$. But this contradicts Theorem 9. Therefore, both sets $BC$ and $BE$ are disjoint from $AC$. It follows that $BC \cap AC = \emptyset$. By definition this means that $A$ and $C$ are on the same side of the line $BD$.

Next, use Axiom K to pick a point $F$ such that $A \rightarrow B \rightarrow F$. Notice that by our Little Lemma that $A$ and $F$ are on opposite sides of $BD$. Since $A$ and $C$ are on the same side of $BD$ it follows by transitivity that $C$ and $F$ are on opposite sides of $BD$. This means that there is an $x \in D \cap BF \cap BD$. Without loss of generality, we may assume that $C \rightarrow x \rightarrow F$ and $x \in BD$. We can consider two cases:

Case 1: Either $B \rightarrow x \rightarrow D$ or $B \rightarrow D \rightarrow x$.

In this case and by our Little Lemma $x$ and $D$ are on the same side of $BC$. Another application of our Little Lemma yields that $x, F$ are on the same side of $BC$. Therefore, by transitivity $D$ and $F$ are on the same side of $BC$. We are given that $A$ and $D$ are on the same side of $BC$ ($D$ is in the interior of $\angle ABC$). Consequently, $F$ and $A$ are on the same side of $BC$. But our Little Lemma tells us that $A$ and $F$ are on opposite sides of $BC$, a contradiction.

Case 2: $x \rightarrow B \rightarrow D$.

So $x$ and $D$ are on opposite sides of $AB$ (Little Lemma). Since $C \rightarrow x \rightarrow F$, $C$ and $x$ are on the same side of $AF\rightarrow AB$. Therefore, by transitivity $C$ and $D$ are on opposite sides of $AB$. This contradicts that $D$ is in the interior of $\angle ABC$.

In both cases we arrived at contradictions. Therefore, our original assumption that $AC \cap BD = \emptyset$ is false and so $AC \cap BD \neq \emptyset$. ■

Theorem 2 [Theorem on Interiors of Angles] Suppose $D, E, F$ are non-collinear and $G$ is on the same side of $DE$ as $F$. If $D$ and $G$ are on opposite sides of $EF$, then $F$ is in the interior of $\angle DEG$.

Proof: To show that $F$ is in the interior of $\angle DEG$ we need to prove that 1) $F$ and $D$ are on the same side of $EG$, and 2) $F$ and $G$ are on the same side of $ED$. Notice that 2) is given in the hypothesis so it suffices to demonstrate 1). So let’s prove 1).

Since $D$ and $G$ are on opposite sides of $EF$. By definition this means that $DG \cap EF \neq \emptyset$. Let $x$ be the point of intersection. Since $x \in EF$, either $x \rightarrow E \rightarrow F$, $E \rightarrow x \rightarrow F$, or $E \rightarrow F \rightarrow x$.

Claim: $x \rightarrow E \rightarrow F$ is false.

Otherwise, by the Little lemma $x$ and $F$ are on opposite sides of $ED$. We are given that $F$ and $G$ are on opposite sides of $ED$ so that by transitivity $G$ and $x$ are on opposite sides of $DE$. But since $D \rightarrow x \rightarrow G$ the Little Lemma applies and yields that $x$ and $G$ are on the same side of $DE$, a contradiction.

Moving on we know that either $E \rightarrow x \rightarrow F$ or $E \rightarrow F \rightarrow x$. In either case, we can apply the Little Lemma so that it is true that $x$ and $F$ are on the same side of $EG$. Since $D$ and $x$ are on the same side of $EG$, transitivity tells us that $F$ and $D$ are on the same side of $EG$, proving 1). ■
Theorem 3  Suppose □ABCD is a Saccheri quadrilateral. Let M and N be the midpoints of BC and AD, respectively. Then MN⊥BC and MN⊥AD. Furthermore, MN≤AB and MN≤CD.

Proof: Since □ABCD is a Saccheri quadrilateral we are given that AB≡CD. By Theorem 57 ∠A ≡ ∠D. By the definition of a midpoint we know AN≡DN and BM≡CM. By (SAS) △ANB ≡ △DNC and so by CPCTC we can say that BN≡NC. By (SSS) △BNM ≡ △CMN and since they are supplementary ∠BNM is a right angle. It follows that MN⊥BC. A similar proof yields that MN⊥AD.

Next, by Corollary 28, we know that ∠A≤90. Thus, ∠A≤∠MNA. Since in □ABMN both ∠B and ∠M are right angles we may apply Theorem 64 to conclude that MN≤AB. ■

Theorem 4  Suppose □ABCD is a Saccheri quadrilateral. Then ℓ(BC) ≤ ℓ(AD). Furthermore, ∠BAC° ≤ ∠ACD°.

Proof: Given the Saccheri quadrilateral □ABCD let M and N be the midpoints of BC and AD, respectively. By the previous theorem we know that □ABNM is a Lambert quadrilateral. Turning the quadrilateral on its side to get □BMNA we know that ∠M and ∠N are right angles and therefore we may apply Theorem 64. Since ∠A ≤ ∠B it follows that BM≤AN. Then

\[ ℓ(BC) = ℓ(BM) + ℓ(MC) = 2ℓ(BM) ≤ 2ℓ(AN) = ℓ(AN) + ℓ(ND) = ℓ(AD), \]

whence BC≤AB.

If ∠ACD < ∠BAC, then by Theorem 45, AD<BC which we just proved to be false. Therefore, ∠BAC° ≤ ∠ACD°. ■

Lemma 2 (Lemma on Defects)  Consider the triangle △ABC. Let D be any point such that A–D–B. Then

\[ δ(△ABC) = δ(△ABD) + δ(△ADC). \]
Theorem 5 The following statements are logically equivalent.

1. Euclid’s 5th Postulate.
2. Playfair’s Parallel Postulate.
3. Hilbert’s Parallel Postulate.
4. If two lines are parallel, then the two lines cut by any transversal have a pair of congruent alternate interior angles.
5. Given two lines $l_1$ and $l_2$ cut by the transversal $l$ satisfy that $l_1 \parallel l_2$, $l \perp l_1$, then $l \perp l_2$.
6. The angle sum of every triangle is 180.
7. There is a triangle whose angle sum is 180.
7’. There is a right triangle whose angle sum is 180.
8. Every Saccheri quadrilateral is a rectangle.
8’. Every Lambert quadrilateral is a rectangle.

Proof:

1. $\Rightarrow$ 4. Suppose $l_1 \parallel l_2$ and they are cut by the transversal $l$. Let $P$ be the point of intersection of $l$ and $l_1$. Suppose $\angle 1$ and $\angle 3$ are remote interior angles and $\angle 2$ is supplementary to $\angle 3$. If $\angle 1 < \angle 3$, then $\angle 1^\circ + \angle 2^\circ < \angle 3^\circ + \angle 2^\circ = 180$. However, by Euclid’s 5th postulate $l \parallel l_2$. therefore, $\angle 1 \not\equiv \angle 3$. A similar argument yields that $\angle 3 \not\equiv \angle 1$, whence $\angle 1 \equiv \angle 3$.

4. $\Rightarrow$ 5. The hypothesis of 4. is the same as the hypothesis of 5. and so the two alternate interior angles are congruent, one of which is a right angle. Therefore, the other is also a right angle.

5. $\Rightarrow$ 2. Suppose $P \not\in l$. By Theorem 31 there is a unique line, say $m$ through $P$ with $m \perp l$. By Theorem 39, there is at least one line, say $l_1$ through $P$ which is parallel to $l$. By 5. we know that $l_1 \perp m$. Also, by 5. if $l_2$ is through $P$ and parallel to $l$ then $l_2 \perp m$. Let $x$ be the point of intersection of $l$ and $m$. If $l_1 \neq l_2$ then we may choose two points $A$ and $B$, both different than $P$, such that $A \in l_1$, $B \in l_2$, and $A$ and $B$ are on the same side of $l$. Since $l_1 \neq l_2$, one of $\angle APX$ and $\angle BPX$ must be bigger than the other. But according to 5. they are both right angles.


6. $\Rightarrow$ 7. Easy.

7. $\Rightarrow$ 7’. Let $\triangle ABC$ be a triangle whose defect is 0. We know that at least two of the angles of $\triangle ABC$ are acute, say $\angle A$ and $\angle B$. Let $D$ be the point such that $\overline{AD}$ and $\overline{CD} \perp \overline{AB}$. By the Lemma on Defects

$$\delta(\triangle ABC) = \delta(\triangle ABD) + \delta(\triangle ADC).$$

Since the defect of a triangle is non-negative and the left-hand side is 0 it follows that $\delta(\triangle ADC) = 0$; a desired right triangle with angle sum equal to 180.

7’. $\Rightarrow$ 9. Let $\triangle ABC$ be a right triangle with right angle $\angle E$. Let $\square ABCD$ be a rectangle with $BC > EF$ and $CD > GE$. By the Ruler axiom there are points $F’$ and $G’$ such that $B - F’ - C$, $C - G’ - D$, $F’C \equiv FE$, and $G’C \equiv GE$. (SAS) implies that $\triangle G’CF’ \equiv \triangle GEF$. Since $\square ABCD$ is Saccheri it is straightforward to show that the defect of $\triangle DCB$ equals 0. By the Lemma on Defects the defect of $\triangle DCB^\circ$ is 0. Applying the Lemma on Defects one more time we arrive at the conclusion that the defect of $\triangle G’CF’$ and hence of $\triangle GEF$ is 0. Therefore, every right triangle has defect 0. Given an
arbitrary triangle we can drop a perpendicular and get that the triangle is made up of 2 right triangles both of defect 0. One last time, the Lemma on Defects tells us that the defect of the arbitrary triangle is 0.

6. ⇒ 8. Since every Saccheri quadrilateral is convex it follows that angle sum of the interior angles of a Sachcheri quadrilateral is the same as the sum of the angle sums of two triangles. Therefore, by 6. the angle sum of the given Sachcheri quadrilateral is 180+180=360 and so it is a rectangle.

8. ⇒ 8’. We assume that every Saccheri quadrilateral is a rectangle and let □ABCD be an arbitrary Lambert quadrilateral. Without loss of generality ∠A has angle measure no greater than 90. We assume ∠A is acute. By Theorem 64, CD<AB. There is a point X between A and D such that BX≡CD. Then X is in the interior of ∠ADC and so ∠XDC < ∠ADC. But □XBCD is a Saccheri quadrilateral and so by hypothesis it is a rectangle. Thus, we arrive a contradiction; both ∠XDC and ∠ADC are right angles.

8’. ⇒ 9. Since a Lambert quadrilateral exists 8’ implies that a rectangle exists.

At this point we have shown that 1. through 5. are all equivalent, while 6. through 9. are all equivalent.

4. ⇒ 6. Given △ABC let ℓ be a line though A which is parallel to →BC. Label the angles as follows ∠A◦ = 1◦, ∠B◦ = 2◦, and ∠C◦ = 3◦. The supplement to 3◦ on the same side of A is 4◦ while the supplement to B on the same side as A is 5◦. Let 6◦ be an interior to 4◦, and 7◦ be an interior to 5◦. By hypothesis 1◦+6◦=5◦, while 3◦=6◦. We know 2◦+5◦=180 so that

\[ 1° + 2° + 3° = 1° + 180 − 5° + 3° = 1° + 180 − (1° + 6°) + 3° = 180 − 6° + 3° = 180. \]

Consequently, the angle sum of every triangle is 180.

Remark 3 To prove that the second set of equivalent statements implies the first set of equivalent we will need to use the following Lemma.

Lemma 4 Assuming 6. Let P ∉ ℓ and let m be the unique line through P such that m ⊥ ℓ. Let S be any point such that ∠SPQ◦ < 90. Then there is a point R on ℓ on the same side as S such that ∠PRQ◦ < ∠SPQ◦.

Proof: Let R be any point on ℓ on the same side of m as S. If ∠PRQ◦ < ∠SPQ◦ then we are done. So we might as well assume ∠PRQ◦ ≥ ∠SPQ◦. By the Ruler Axiom pick a point R’ so that Q–R–R’ and RR’≡PR. △PRR’ is an isosceles triangle. Notice that by 6.

\[ 180 = 2∠PR’R◦ + ∠PRR’◦ = 2∠PR’R◦ + 180 − ∠PRQ◦. \]

it follows that ∠PR’Q◦ = \( \frac{1}{2} \)∠PRQ◦. Repeating this process and appealing to the Archimedean property of the real numbers it follows that eventually there is a point R such that ∠PRQ◦ < ∠SPQ◦.

6. ⇒ 2. Let P ∉ ℓ. let M be the unique line through P which is perpendicular to ℓ. We know that there is at least one line through P, say ℓ₁, which is perpendicular to m, and hence ℓ ∥ ℓ₁.