

We want to examine the quantity

$$p(x, t) = \sum_{n=1}^{\infty} p_n(x, t) = \sum_{n=1}^{\infty} b_n(0)b_n(x)T_n(t), \quad (1)$$

where

$$b_n(x) = \sqrt{\frac{2}{a+b}} \sin\left(\frac{n\pi(x+b)}{a+b}\right) \quad (2)$$

and

$$T_n(t) = \exp\left(\frac{-n^2\pi^2\sigma^2t}{2(a+b)^2}\right). \quad (3)$$

As  $n$  grows, the terms  $T_n(t)$  become very small very fast, which implies that the sum in (1) can be well-approximated with by retaining only the first  $N$  terms, say, with  $N = 10$ .

The family of functions  $\{b_n\}$  forms an orthonormal basis for  $L^2(-b, a)$ .

Each  $p_n$  satisfies

$$\frac{\partial p_n}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p_n}{\partial x^2} \quad (4)$$

$$p_n(-b) = 0 \quad (5)$$

$$p_n(a) = 0. \quad (6)$$

The function  $p$  given in (1) satisfies these conditions and additionally,  $p(\cdot, 0) = \delta$  (the Dirac distribution).

As discussed elsewhere,

$$f(x, a, b) \stackrel{\text{def}}{=} \frac{\partial^2 p(x, 1)}{\partial a \partial b}$$

is the joint PDF for intraday high, low, and closing price of a stock following GBM. This is the quantity we want to examine.

Related:

$$g(a, b) = \int_{x=-b}^a f(x, a, b) dx$$

is the joint high-low PDF, so the *conditional* PDF ...

$$\phi(x|(a, b)) = f(x, a, b)/g(a, b)$$

**EXERCISE:**

With  $a = 1/2$ ,  $b = 1$ , and  $\sigma = 1$ , make a plot of  $\phi(x)$ ,  $-b \leq x \leq a$ .