Research Statement

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1 Introduction

I study the characters and class functions of certain groups, which in the literature are variously called adjoint groups, algebra groups, circle groups, or quasi-regular groups. Computing the irreducible characters of such groups is commonly viewed as a difficult problem, and various stand-ins for the irreducible characters have been suggested. I have been examining what underlying structure is revealed by two of these stand-ins, the supercharacters and the Kirillov functions, and I have been studying how they interact with each other and with the irreducible characters.

Since graduating, I have also begun to collaborate with a colleague on an examination of the lattice-ordered group consisting of the space $\mathrm{Lip}(Z)$ of Lipschitz functions from a metric space Z to $\mathbb{R}$. The bulk of the material below will explore characters, but I will briefly touch on this collaboration in the final section.

2 History

The power and versatility of character theory have been amply demonstrated. A very brief summary follows for those who are not familiar with the subject. A character is a specific kind of map from a group to the complex numbers. Although some of the structure of the group is lost in passing from the group to its characters, much is preserved. For example, knowledge of the characters allows one to recover the order, the normal subgroups, the commutator subgroup, and other important features of the group.

A variety of applications call for determining the character table of $U_n(F)$, the group of upper-triangular $n \times n$ matrices over a finite field F with diagonal entries 1. Unfortunately, there are no known general descriptions for these character tables. Kazhdan was able to calculate the irreducible characters of $U_n(F)$ in the case when the characteristic of F was large [3]. Kirillov modified the resulting functions to apply to arbitrary characteristic [4], but they lost some of their effectiveness. André utilized Kazhdan's description to create the first supercharacters [1], and Yan presented an approach more general than that of André [7]. Diaconis and Isaacs further generalized this to arbitrary algebra groups [2], and my work continues within this general context.

3 Preliminaries

Let J be a finite-dimensional, nilpotent algebra over a finite field F. The set

$$1 + J = \{1 + x \mid x \in J\}$$

forms a group with multiplication inherited from J; a group of this form is called an algebra group. Let $G = 1 + J$.

Let J^* denote the dual space of J, the set of F-linear maps from J to F. There are natural left, right, and conjugation actions of G on J^*. For $\lambda \in J^*$, write $\mathcal{O}_\lambda$, λG, and $G\lambda G$, respectively, for the conjugation, right, and two-sided orbits of λ. Fix a non-trivial group homomorphism φ from the additive group of F to $\mathbb{C}$. The Kirillov function ψ_λ is defined by

$$\psi_\lambda(1+x) = \frac{1}{\sqrt{|\mathcal{O}_\lambda|}} \sum_{\mu \in \mathcal{O}_\lambda} \varphi(\mu(x))$$

and the supercharacter χ_λ is defined by

$$\chi_\lambda(1+x) = \frac{|\lambda G|}{|G\lambda G|} \sum_{\mu \in G\lambda G} \varphi(\mu(x)).$$

The Kirillov functions are an orthonormal basis for the space of class functions of G, but they need not be characters. The supercharacters are a set of mutually orthogonal characters, but they need not span the class functions. Thus, each set of functions has its own strengths and weaknesses.

4 Results

We begin by sampling some elementary results demonstrating ways in which the theory of supercharacters mimics the general theory of characters. As before, let F be a finite field.

Proposition 4.1 *Let J and K be finite-dimensional, nilpotent F-algebras.*

- (i) *For each ideal I of J, there is a collection of supercharacters whose kernels intersect to give exactly $1 + I$.*
- (ii) *A surjective algebra homomorphism from J to K yields a natural map between the supercharacters of J and the supercharacters of K.*
- (iii) *The supercharacters of J are all linear if and only if $J^2 = 0$.*
- (iv) *If A is a subalgebra of J such that $A^2 = 0$, then the degree of every supercharacter of J is bounded by $(|J|/|A|)^2$.*

There are many theorems in group theory concerning character degrees and conjugacy class sizes. Supercharacters yield some similar results. We call the degree of a supercharacter a superdegree, and the supercharacter analog of conjugacy classes are called superclasses. Specifically, the superclass of the group element $1 + x$ in the algebra group $1 + J$ is the set $\{1 + gxh \mid g, h \in 1 + J\}$. It can be shown that the set of superdegrees and the set of superclass sizes both must be powers of $|F|$ that contain 1. The following result gives the converse.

Theorem 4.2 *Let $\mathcal{S}$ denote a set of powers of $|F|$ that contains* 1.

(i) *There is an algebra whose set of superdegrees is exactly $\mathcal{S}$.*

(ii) *There is an algebra whose set of superclass sizes is exactly $\mathcal{S}$.*

The next result has no analog in ordinary character theory. An immediate corollary is that the nilpotence class of an algebra group is bounded by one more than the number of superdegrees.

Theorem 4.3 *Let J be a finite-dimensional, nilpotent algebra.*

(i) *If J has exactly n different superdegrees, then $J^{n+1} = 0$.*

(ii) *If J has exactly n different superclass sizes, then $J^{n+1} = 0$.*

Since Kirillov functions and supercharacters are class functions, they can be decomposed into irreducible characters. The following results demonstrate the relationship between all three of these kinds of functions.

Theorem 4.4 *Two Kirillov functions that share a linear constituent must arise from functionals in the same two-sided orbit.*

Theorem 4.5 *Every irreducible constituent of a Kirillov function is also a constituent of the supercharacter arising from the same functional.*

Theorem 4.5 says that decomposing a supercharacter into its Kirillov functions and then decomposing the resulting Kirillov functions into their irreducible constituents is the same as passing straight from the supercharacter to its irreducible constituents. This correspondence allows one to utilize the respective strengths of one set of functions even while dealing with the other.

5 Future Research

I am currently expanding Theorem 4.2. More interestingly, it remains to discover how constituents of the same Kirillov function are related to each other; computer calculations hint at a very striking structure that has yet to reveal itself. Kazhdan's original construction relied on utilizing the structure of the underlying algebra to define a function that behaves like a logarithm. This function allows one to carry the easily described character structure of the additive group of J to the algebra group $1 + J$. Even when this logarithm can not be defined, however, the construction often yields the irreducible characters; there is, as yet, no explanation for this good fortune. The story recently became a little less confusing when I was able to show that introducing any logarithm-like function only permutes the supercharacters. Thus, the role of the logarithm-like functions seems to be to push the Kirillov functions toward being characters. The supercharacters, which are already characters, remain unaffected by any such attempts.

Finally, the aforementioned collaboration concerning the lattice-ordered group Lip(Z) is progressing. Borrowing from the similar ring theoretic concept (see [6]), Hager, Kimber, and McGovern have recently defined when a lattice-ordered group is clean. It currently looks as if it will be possible to characterize precisely when Lip(Z) is clean; this problem is motivated by a similar characterization for clean spaces of continuous functions from a topological space to $\mathbb{R}$ (see [5]).

References

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